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External vs. Internal Didactic Transposition of the Definition of Limit in Calculus

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ABSTRACT

The limit of a function is a foundational concept in calculus, yet its transition from scholarly knowledge into knowledge to be taught, taught knowledge, and learned knowledge may produce shifts in meaning that affect students' conceptual understanding. Limited studies have examined how these shifts occur through external and internal didactical transposition. This study examines how the meaning of the limit definition is transformed through these stages and how such transformations are reflected in students' understanding. A qualitative approach was employed through document analysis of calculus textbooks, analysis of students' lecture notes to reconstruct the lecturer's taught knowledge, and analysis of students' responses to a diagnostic task. Using purposive sampling, 25 undergraduate students enrolled in a Calculus course were selected as participants. The findings show that, through external didactical transposition, the formal ε–δ definition is simplified into intuitive and procedural forms in textbooks, thereby reducing its epistemological scope. Through internal didactical transposition, this simplification is further reinforced in the taught knowledge reconstructed from students' lecture notes, which predominantly emphasize algebraic manipulation and direct substitution. Consequently, students' understanding is expressed through three main response patterns: interpreting the limit as approaching a value, applying procedural techniques, and equating the limit with direct substitution. These findings reveal a shift from the formal mathematical meaning of the limit toward intuitive and procedural interpretations. The study highlights the need for didactical designs that explicitly connect intuitive representations and procedural techniques with the formal ε–δ definition.

Keywords: External Didactic Transposition, Internal Didactic Transposition, Limit of Function, Calculus Learning, Knowledge Transformation

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INTRODUCTION

The concept of limit is a foundational element of calculus and mathematical analysis, serving as the epistemological basis for continuity, differentiation, and integration. Historically, the formalization of the limit concept particularly through the ε–δ definition that marked a decisive moment in the development of mathematical rigor, resolving longstanding ambiguities associated with intuitive and infinitesimal reasoning (Grabiner, 1983; Katz & Sherry, 2013). Despite its centrality, research over several decades has consistently shown that students struggle to develop a coherent and theoretically grounded understanding of limits, even after formal instruction (Tall & Vinner, 1981; Cornu, 1991; Williams, 2001; Tall, 2004; Sebsibe & Feza, 2020; Baye et al., 2021).

These difficulties are not merely individual cognitive shortcomings but are deeply connected to the institutional transformation of mathematical knowledge. The theory of didactical transposition, originally introduced by Chevallard (Chevallard, 2019), provides a powerful framework for analyzing how scholarly mathematical knowledge is transformed as it enters educational systems. Didactical transposition describes the process through which scholarly knowledge is transformed into knowledge to be taught, then into taught knowledge, and eventually into learnt knowledge. This process is shaped by curricular structures, institutional constraints, pedagogical norms, and assessment practices (Chevallard, 1991; Winsløw, 2007).

Within the framework of didactical transposition, the transformation of mathematical knowledge occurs through two main processes, namely external and internal didactic transposition. External didactic transposition refers to the transformation of scholarly knowledge into knowledge to be taught, typically mediated by curriculum design, textbooks, and educational policies (Chevallard, 1991). In contrast, internal didactic transposition takes place at the classroom level, where teachers interpret and enact this knowledge into taught knowledge through instructional strategies, representations, and pedagogical interactions (Brousseau, 1997; Chevallard, 1991). Distinguishing between these two forms of transposition is essential for understanding how the meaning of mathematical concepts, such as limits of functions, can shift from their formal structure to more contextualized classroom interpretations, which may lead to gaps in students' understanding if not carefully managed didactically.

In the case of the limit concept, didactical transposition often involves a substantial shift of meaning. The scholarly definition of limit is characterized by logical precision, nested quantifiers, and relational dependency between variables. However, within instructional contexts, this formal structure is frequently replaced by intuitive metaphors such as "approaching," "getting closer," or "tending to a value," often supported by graphical or numerical representations (Cornu, 1991; Tall, 1993; Artigue, 2000). While these approacheature of the limit from a formally defined mathematical object into a dynamic or perceptual process.

This transformation is particularly problematic when intuitive representations are not explicitly connected to the formal definition. According to Tall and Vinner (Tall & Vinner, 1981) that distinction between concept image and concept definition, students tend to construct personal mental images of limits based on instructional experiences. When didactical transposition privileges intuitive and procedural aspects over formal reasoning, students' concept images may diverge significantly from the mathematical concept definition, leading to stable learning obstacles (Fischbein, 1987; Tall, 2004) such learning obstacles often persist into advanced mathematics and hinder students' ability to engage in formal proof and abstract reasoning (Williams, 2001; Tall & Katz, 2014)

Although previous studies have documented students' intuitive and procedural difficulties with limits (Cornu, 1991; Tall, 2004; Tall & Vinner, 1981; Williams, 2001), limited research has traced how these difficulties emerge as the initial definition of limit is transformed across scholarly knowledge, knowledge to be taught, taught knowledge, and learned knowledge. This issue is important because didactical transposition necessarily involves selecting, reorganizing, and simplifying scholarly knowledge to make it teachable and accessible (Chevallard, 1991; Chevallard & Bosch, 2020). However, excessive simplification or omission of essential meanings may create an epistemological distance between students' understanding and the underlying mathematical concept, resulting in an intuitive or operational conception that is insufficiently connected to its formal and structural meaning (Sfard, 1991; Tall & Vinner, 1981). Such distortions may remain unnoticed, persist into subsequent calculus topics, and hinder students' formal mathematical reasoning. Therefore, this study examines how the initial definition of limit is transformed through external and internal didactical transposition and how these transformations are reflected in students' understanding, with the aim of maintaining a balance between pedagogical accessibility and epistemological integrity.

Another major consequence of didactical transposition is the proceduralization of the limit concept. Numerous studies have shown that instructional practices frequently emphasize algebraic techniques for evaluating limits, such as substitution, factorization, or rationalization, without adequately addressing the conceptual foundations that justify these techniques (Sfard, 1991; Artigue, 2000; Nardi, 2008). Although procedural fluency may serve as an important precursor to structural understanding, difficulties arise when students remain at the operational level and are not supported in progressing toward a structural conception of limits (Sfard, 1991). This tendency was also identified by Tonra et al. (2025), who found that students' procedural ability to solve limit problems did not necessarily correspond to an understanding of the formal definition, which remained their most prominent learning obstacle.

Textbooks and curriculum documents play a decisive role in stabilizing these transformed meanings. This is particularly significant because the formal ε–δ definition was developed to resolve historical ambiguities concerning the uncertain status of infinitesimals and the absence of a precise quantitative meaning of a variable or function value "approaching" a particular value (Grabiner, 1983; Katz & Sherry, 2013). However, analyses of limit instruction and widely used calculus textbooks indicate that the formal definition is often introduced after informal treatments and may remain weakly connected to computational tasks, which tend to emphasize routine limit evaluation rather than connections between intuitive and formal meanings (Barbé et al., 2005; Hong, 2023; Viirman et al., 2022). Such institutional choices contribute to the institutionalization of knowledge, whereby specific meanings become dominant and resistant to change within educational systems (Chevallard, 1991).

To systematically investigate these transformations, didactical transposition comprises two interconnected stages: external transposition, which occurs within the noosphere beyond the classroom, and internal transposition, which takes place through classroom practices (Chevallard, 1991; Winsløw, 2011). In this study, the noosphere is represented by the institutional processes underlying the study-program curriculum such as Semester Learning Plan, and calculus textbooks, through which scholarly knowledge is selected and organized as knowledge to be taught. The knowledge to be taught is further adapted by lecturers during classroom instruction, becoming taught knowledge, shaped by pedagogical strategies, time constraints, and assumptions about students' prior knowledge (Brousseau, 1997; Nardi, 2008).

Internal transposition occurs when lecturers interpret and present this planned knowledge in classroom instruction, while students subsequently construct it as learned knowledge according to their understanding and individual characteristics. Through these processes, didactic transposition makes knowledge more structured, accessible, and pedagogically appropriate than its original scholarly form (Chevallard & Bosch, 2020). The final stage of this transposition concerns classroom interactions in which students actively construct meaning, resulting in learnt knowledge as the outcome of learning processes (Chevallard, 1991). At this stage, discrepancies between scholarly knowledge and students' understanding become visible, revealing learning obstacles that are rooted in earlier stages of transposition (Brousseau, 1997; Cornu, 1991).

To identify these obstacles empirically, this study employs a diagnostic test focusing on the limit of functions. Previous studies have examined students' understanding of limits using diagnostic tasks informed by the concept image–concept definition distinction, APOS theory, and epistemological-obstacle frameworks (Tall & Vinner, 1981; Cottrill et al., 1996; Adhikari, 2020; Sebsibe & Feza, 2020; Sulastri et al., 2022). The results of this diagnostic test provide evidence of students' conceptual difficulties and meaning shifts. These findings serve as the foundation for critical transposition, in which the researcher examines how specific didactical choices contribute to observed obstacles and proposes alternative didactical designs aimed at restoring epistemological coherence and conceptual depth (Chevallard, 2019; Artigue, 2014).

Understanding didactical transposition is essential because it reveals how mathematical knowledge is transformed from scholarly production into curriculum, classroom instruction, and student learning. Without examining these transformations, many learning obstacles whether epistemological, ontogenic, or didactical that are often misattributed solely to students' cognitive limitations rather than to institutional and pedagogical decisions that shape what is taught and how it is taught (Brousseau, 1997; Chevallard, 1991). By making the stages of transposition explicit, educators and researchers can identify how epistemological distortions, inappropriate sequencing, or excessive procedural emphasis generate specific learning obstacles in students' understanding (Cornu, 1991; Artigue, 2000; Tall, 2004).

Consequently, investigating didactical transposition is crucial for diagnosing the roots of learning obstacles and for designing instruction that maintains conceptual rigor while supporting meaningful and accessible mathematical learning (Chevallard & Bosch, 2020; Artigue, 2014).

By integrating didactical transposition theory with a detailed analysis of the limit concept, this study seeks to clarify how institutional transformations shape mathematical meaning and influence students' understanding. Such an analysis is crucial for addressing the enduring tension between mathematical rigor and pedagogical accessibility and for designing instruction that supports both intuitive insight and formal reasoning in calculus education.

In order to examine the transformation of mathematical knowledge across institutional levels, this study addresses the following research question: 1) What is the nature of the limit definition as scholarly knowledge in Calculus? 2) How is scholarly knowledge transformed into knowledge to be taught through external didactic transposition? 3) How is knowledge to be taught transformed into taught and learnt knowledge through internal didactic transposition? 4) How are the relationship among the knowledge to be taught, the taught knowledge, and the learned knowledge of the limit definition?

METHOD

Research Design

This study employed a qualitative case study design to investigate the transposition of knowledge concerning the concept of limit within a bounded educational context (Creswell & Poth, 2018). This study employed a qualitative case study design within a third-semester Calculus course (Creswell & Poth, 2018). Didactical transposition was used to examine changes in the concept of limit across scholarly knowledge, knowledge to be taught, taught knowledge, and learned knowledge (Chevallard & Bosch, 2014). Scholarly knowledge was identified from authoritative mathematical literature, while knowledge to be taught was examined through the curriculum, Semester Learning Plan, and calculus textbooks. Taught knowledge was reconstructed from students' lecture notes and confirmed through a lecturer interview. Learned knowledge was identified from students' diagnostic-test responses. The four stages were then compared to identify changes, simplifications, and shifts in meaning during the transposition process. The process is illustrated in the following figure 1

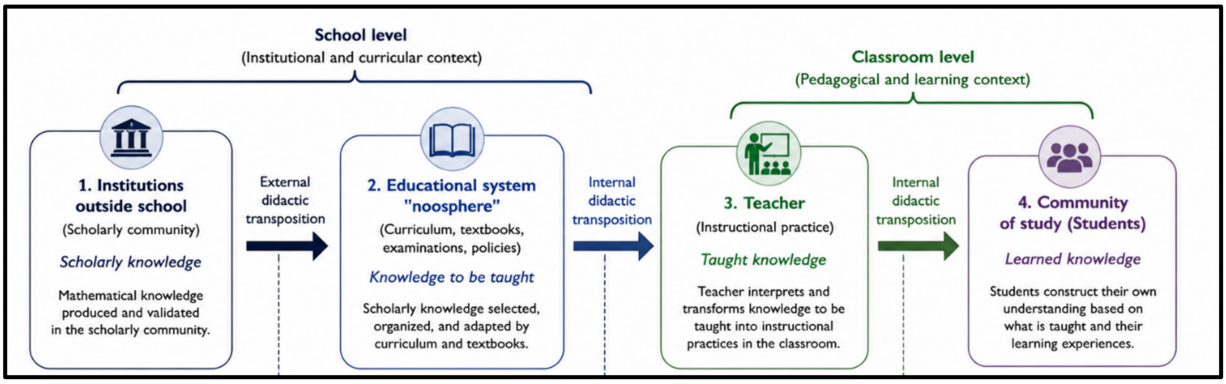


Figure 1. The didactic transposition process adopted from Chevallard and Bosch (2014)

Participants

A female mathematics lecturer at Universitas Khairun voluntarily participated in the study. She had four years of experience teaching mathematics and was the lecturer responsible for the selected Calculus course. The student participants consisted of 25 third-semester students enrolled in the Calculus class during the 2025/2026 academic year. They were selected through criterion-based purposive sampling. The student inclusion criteria were as follows:

1. Official enrollment in the selected third-semester Calculus class.
2. Completion of the limit of functions topic in the Differential Calculus course.
3. Attendance at the relevant classroom sessions and availability of lecture notes that could be used to reconstruct the taught knowledge.
4. Willingness to complete the diagnostic test and participate in the study.

Instruments and Data Collection Procedures

This study was conducted in several phases. In the first phase, the aim was to analyze several major Calculus textbooks. The researchers examined various expert definitions of limits. There are seven definitions that are discussed in this paper as a starting point for understanding the concept of limits. These were then compared with the Calculus textbook used by the lecturer. The textbook used was written by Purcell, which is very well known in Indonesia and is used by most Calculus lecturers in the country.

In the second phase, all students' lecture notes were collected and initially reviewed to identify recurring content related to the limit concept. Three notes were then selected for more detailed comparison based on the completeness of the recorded definitions, examples, procedures, and lecturer explanations, rather than on the correctness of the students' mathematical work. The similarities and differences identified across the notes were used to reconstruct the main knowledge presented during classroom instruction. Because direct classroom observation was not feasible due to the mismatch between the research timeline and the teaching schedule, the reconstruction of taught knowledge from students' lecture notes was verified through a semi-structured interview with the lecturer. The interview examined whether the definitions, examples, representations, procedures, and instructional sequence recorded in the students' notes were consistent with the lecturer's account of the lesson. The interview was also used to clarify any content or instructional purposes that were not fully documented in the notes.

Next, the researchers administered a diagnostic test to examine the extent of students' understanding of limits. A focus group discussion with three experts in calculus and mathematics education was conducted to qualitatively review the diagnostic-test items. The experts examined the relevance, clarity, mathematical accuracy, and alignment of the items with the research objectives. Therefore, no quantitative validity index was calculated. After that, a pilot study involving three third-semester students was carried out, and several items were revised to make them clearer and more understandable based on student responses. For example, the initial question, "Explain your understanding of the meaning of a limit," was considered too general and was revised to "Explain your understanding of the meaning of the limit for f(x). Thus, the draft diagnostic test was ready to be administered to one third-semester Calculus class in this study.

The complete diagnostic test consisted of seven open-ended items examining different aspects of limits, including the meaning of a limit, limits at infinity, and continuity. However, this article focuses specifically on students' understanding of the meaning of a limit; therefore, only the item directly related to this aspect was analyzed and reported.

Table 1. The Limit of Function Diagnosis Test

The Indicators	Instrument Test
Understanding the meaning of limits of functions	Given the function $f(x) = \frac{x^2-1}{x-1}$ a. Finding the indicate limit $\lim_{x\to 1} \frac{x^2-1}{x-1}$ b. Explain in your own words the meaning of limit for $f(x)$

The 25 written responses were reviewed and grouped into three broader patterns based on similarities and differences in students' interpretations of limits. One representative response was selected from each group and identified as M1, M2, and M3.

Data analysis

Data analysis employed thematic synthesis, proceeding across three stages: (1) descriptive coding of key findings, technology types, deployment contexts, and outcomes per study; (2) comparative analysis across studies to identify interpretive themes representing recurring patterns; and (3) mapping of analytical themes onto the research questions to generate overarching propositions. Descriptive statistics were calculated to characterise the evidence base.

Data credibility

Data credibility was ensured through source and method triangulation. Curriculum documents such as the Semester Learning Plan, and calculus textbooks were compared to analyze knowledge to be taught. The reconstruction of taught knowledge from students' lecture notes was confirmed through an interview with the lecturer. Students' written responses were also compared with their follow-up interview explanations to clarify the meanings underlying their answers.

Ethical Considerations

The lecturer and students signed informed-consent statements confirming their voluntary participation and permission to use their interview responses, written answers, and lecture notes for research purposes. All participant identities were anonymized using codes.

RESULTS & DISCUSSION

This research will look at the shift in knowledge that begins by looking at the definition of limits through scientists, which is known as scholarly knowledge, then continues by examining the definition of limits in textbooks that used directly by lecturers as knowledge to be taught, after that this research continues by looking at how lecturers teach the definition of limits and the last is what the limits meaning understood by the students.

Scholarly knowledge of limit definition

Scholarly knowledge is essential as the epistemological reference point for analyzing didactical transposition. It provides the original mathematical meaning against which knowledge to be taught, taught knowledge, and learned knowledge can be compared. By establishing the intuitive, formal, and more general formulations of limit at the scholarly level, the analysis can identify which mathematical elements are retained, simplified, transformed, or omitted during the transposition process.

There are several definitions of limits from experts in scholarly knowledge, but here the definitions will be grouped into 3 different definitions of limit. The classification into three groups is grounded in differences in the formulations of the limit definitions. The first group of definitions is that of Ellis and Gulick (1982) is similar to the definition of Leithold (1983), Martono (1987), and Stewart (1999). These four authors all state that the domain of f must be defined on an open interval I containing a, but (at x = a, it may be undefined). Therefore, we say that the limit of f(x) as x approaches a is L, and we write $\lim_{x \to a} f(x)$. The second group of definitions is that of Stein (1982), who states that the domain of f contains the open intervals (c, a) and (a, b), where a may or may not be in the interval. The third group of definitions is that of Bartle and Sherbert (1927), that is not specify an open interval. They only state that a is a cluster point or limit point, making the domain set more general.

Knowledge to be taught of limit definition

This definition of the limit is taken from a calculus textbook as knowledge to be taught that has been widely used across Universities in Indonesia, the book from Varberg., Purcell, & Rigdon, 2007. This book is also the only textbook used by the lecturers who teach the Calculus course at Universitas Khairun. In this book, the intuitive and precise definitions of the limit are stated as follows:

İntuitive Definition of Limit: To say that $\lim_{x \to a} f(x) = L$ means that when x near but different from c then $f(x)$ is near L. While the precise meaning of limit: To say that $\lim_{x \to a} f(x) = L$ means that for each given **ε** > 0 (no matter how small) there is a corresponding δ > 0 such that $|f(x) - L| < \varepsilon$, provided that $0 < |x - a| < \delta$; that is,

$$0 < |x - a| < \delta \rightarrow |f(x) - L| < \varepsilon$$

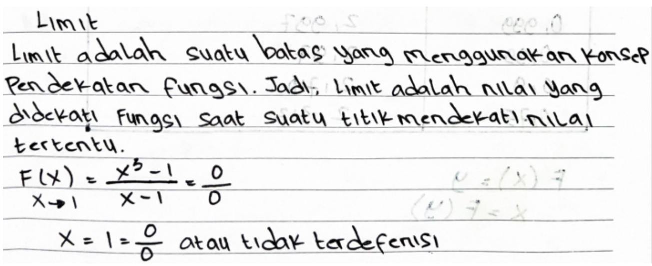
This definition corresponds to the first group of limit definitions identified at the level of scholarly knowledge.

Taught knowledge and learnt knowledge of limit definition

Taught knowledge here is the limit definition that is actually delivered and enacted by the lecturer in the classroom, resulting from the adaptation of the “knowledge to be taught” into real teaching practice (Chevalard & Bosch, 2020). According to the students’ lecture notes, the lecturer begins by directly providing the definition of a limit, stating: "A limit is a boundary that involves the concept of approaching a function. Thus, the limit is the value a function approaches as a variable nears a certain point." The lecturer presents the following function:

$$f(x) = \frac{x^3-1}{x-1}$$

Then, the students are asked to find $f(1)$ which results in $f(1) = \frac{0}{0}$ or undefined. The following picture is lecturer’s explanation taken from students’ lecture notes.



The translation: Limit is a boundary defined through the idea of a function approaching a value. So, limit is the value that a function approaches when a point approaches a certain value.

The lecturer presents the following f(x):

$f(x) = \frac{x^3-1}{x-1} = \frac{0}{0}$

$x = 1 = \frac{0}{0}$ = undefined

Figure 2. Lecturer’s context of limit as a boundary

Analysis of the students’ lecture notes showed that the lecturer initially introduced limit as the value approached by a function as the variable approaches a particular point. The notes documented that the lecturer presented $f(x)\frac{x^3-1}{x-1}$, showed that direct substitution at $x = 1$ produced $\frac{0}{0}$, and then demonstrated algebraically that

$$\lim_{x\to1}\frac{x^3-1}{x-1} = \lim_{x\to1}\frac{(x^2+x+1)(x-1)}{x-1} = \lim_{x\to1} x^2 + x + 1 = 1^2 + 1 + 1 = 3$$

In the interview, the lecturer explained that this example was selected because $f(1)$ was undefined while the limit existed. The lecturer stated that the example was intended to direct students' attention away from the value of the function exactly at $x = 1$ and toward the behavior of $f(x)$ near $x = 1$. Next, the students' lecture notes also documented the use of a numerical table and a graph showing values of x approaching 1 from both sides.

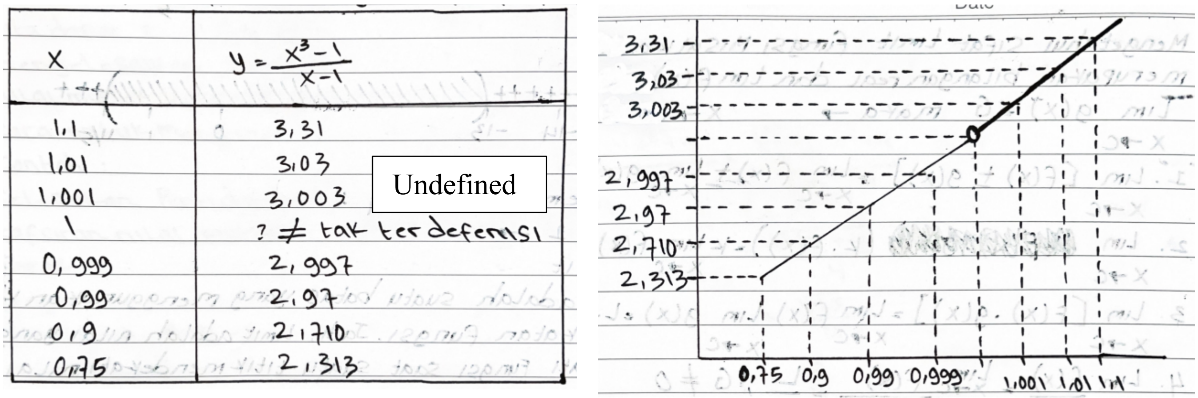


Figure 3. Lecturer's explanation to teach the meaning of limit

In the interview, the lecturer explained that these representations were used to make the approaching process visible and to help students distinguish between the function value at a point and the limit of the function near that point. Thus, the interview explanation clarified the instructional purpose of the table and graph recorded in the students' notes.

The students' lecture notes further showed that the lecturer subsequently introduced the example

$$\lim_{x\to3}(4x-5) = (4.3\text{-}5) = 12\text{-}5 = 7$$

The lecturer then explained that limits can be evaluated using three methods: direct substitution, factorization, and rationalization using the conjugate. The lecturer confirmed during the interview that these procedures were introduced after the intuitive explanation to help students evaluate different forms of limit problems. Comparison of the two data sources showed consistency between the instructional sequence recorded in the students' lecture notes and the lecturer's explanation.

From the data of taught knowledge above, it can be observed that a shift in the definition of the limit has occurred from the *knowledge to be taught* to the *taught knowledge* enacted by the course lecturer. The lecturer initially defines the limit as a boundary based on the idea of a function approaching a value, thereby introducing a boundary-oriented context. Subsequently, however, the lecturer adopts the same definition as that proposed by Varberg et al. (2007).

This study further examines learning obstacles through the analysis of *learnt knowledge* concerning the definition of limit. In the framework of didactical transposition, learnt knowledge refers to the knowledge that students construct after engaging with the knowledge presented by the lecturer. It is not necessarily identical to the taught knowledge because students interpret, reorganize, and understand the instructional content based on their prior knowledge and learning experiences. In this case, the definition of limit that students understand is only limited to the informal definition that is asked, not leading to the formal definition. Therefore, this study aims to explore students' understanding of the concept of the limit of a function through a problem-based task from diagnostic test. There are 3 groups of student answers represented by M1, M2 and M3 in responding to problems in the diagnostic test. Here in table 2, describing the students' understanding of the meaning of limit.

Table 2. The students' understanding of the (intuitive) definition of limit

Code	Students answer	Description
M1	limit adalah suatu fungsi yang mendekati suatu titik. Contoh pada $f(x) = \frac{x^2-1}{x-1}$, kita substitusikan $x = 1$ maka $= \frac{(1)^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$ (tidak terdefenisi) jika kita menggunakan $\lim_{x\to1} \frac{x^2-1}{x-1}$ $\lim_{x\to1} \frac{(x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$ $\lim_{x\to1} x+1$ $= 1+1 = 2$. Jadi, jika kita mensubstitusikan nilai $x = 1$ maka hasilnya tidak terdefenisi dan jika kita menggunakan limit atau x mendekati satu hasilnya dua (2)	Limit is a function that approaches a point. Example: $f(x)=\frac{x^2-1}{x-1}$, when substitute $x = 1$ then $f(x) = \frac{(1)^2-1}{1-1} = \frac{1-1}{1-1} = \frac{0}{0}$ undefined. $\lim_{x\to1}\frac{x^2-1}{x-1} = \lim_{x\to1}\frac{(x-1)(x+1)}{(x-1)} = \lim_{x\to1}(x+1) = 1+1= 2$ So if we subtitute the value of x =1 then the result is undefined and if we use the limit or x approaches one the result is two.
M2	Substitusi $\lim_{x\to1} \frac{x^2-1}{x-1} = \frac{1^2-1}{1-1} = 0$ Faktorisasi $\lim_{x\to1} \frac{x^2-1}{x-1} = \frac{(x+1)\cancel{(x-1)}}{\cancel{x-1}} = x+1 = 1+1 = 2$ Limit yang saya pahami adalah kita bisa mencari limit dengan menggunakan pemfaktoran dan substitusi untuk mencari nilai limitnya	The limit that I understand is that we can find the limit by describing the factorization and substitution to find the limit value. From the interview, M3 said that if $f(x)$ subtitute the value of x =0 then do factorization
M3	Makna limit untuk f(x) merupakan suatu fungsi yang dimana nilai suatu limit akan digantikan oleh nilai yang ada di f(x)	The meaning of limit for $f(x)$is a function where the value of a limit will be replaced by the value in $f(x)$.

The 25 written responses were classified into three groups based on similarities in students' interpretations and solution approaches. Each group was represented by one selected response, identified as M1, M2, and M3. The first group, represented by M1, understood that as x approaches 1, $f(x)$ approaches 2, even though $f(1)$ is undefined. This group used the term "approaches" to explain the distinction between the limit value and the function value at $x = 1$. The second group, represented by M2, determined the limit procedurally through factorization followed by substitution. Their responses emphasized the calculation process, while the conceptual meaning of the limit was not explicitly explained. The third group, represented by M3, treated the limit primarily as the direct substitution of the value of x into the function. Based on the didactical transposition of the limit definition, the findings indicate that students' learned knowledge varied in its distance from scholarly knowledge. While the first group retained the intuitive meaning of approaching behavior, the second and third groups emphasized procedural evaluation. These shifts in formulation and meaning may contribute to learning obstacles that affect students' conceptual development.

The relationship among the knowledge to be taught, the taught knowledge, and the learned knowledge of the limit definition

In this study, the relationship among the knowledge to be taught as formulated in calculus textbooks, the taught knowledge enacted by the lecturer, and the learned knowledge constructed by the students with respect to the definition of limit was examined (see figure 3). The analysis begins with scholarly knowledge, in which the definition of limit is characterized by logical precision and quantified relationships between ε and δ. This scholarly knowledge was then transposed into knowledge to be taught through the calculus textbook by Varberg et al. (2007) which presents both an intuitive definition and a precise ε–δ formulation. This process reflects what Chevallard (2019) describes as external didactic transposition, in which scholarly knowledge is reshaped according to institutional and curricular demands.

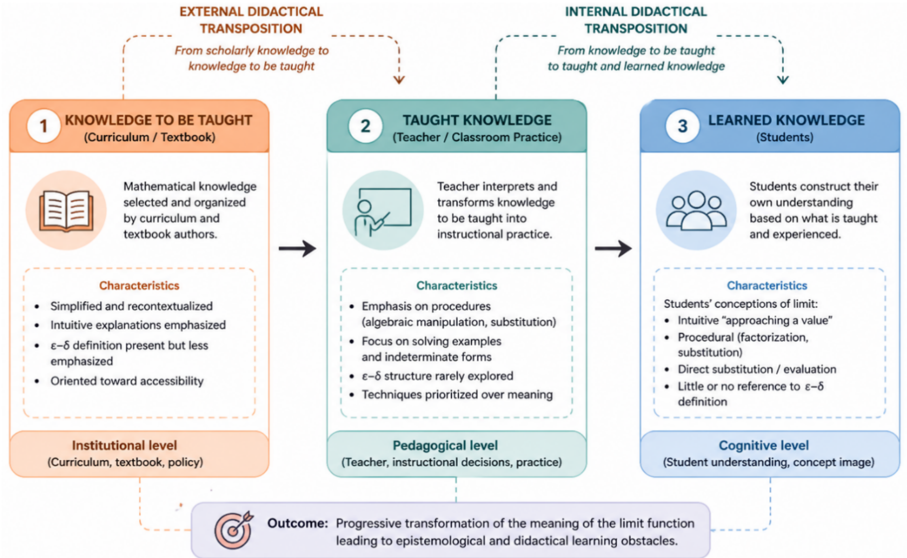


Figure 4. The relationship among the knowledge to be taught, the taught knowledge, and the learned knowledge

In classroom practice, however, the lecturer's taught knowledge emphasized mainly the intuitive and procedural aspects of the limit. The lecturer introduced the limit as "a boundary involving the idea of approaching," and focused on evaluating limits using algebraic manipulation and substitution, illustrated through the example $f(x) = \frac{x^3-1}{x-1}$, students are asked to find $f(1)$ which results in $f(1) = \frac{0}{0}$ or undefined were then guided to factor and simplify the expression in order to obtain the limit value. Although the lecturer adopted the same formal definition as in Varberg et al. (2007), in practice the teaching foregrounded techniques rather than the relational meaning of ε–δ. This reflects what Sfard (1991) and Artigue (2000) describe as the proceduralization of mathematical concepts in instruction.

The learned knowledge of the students, as revealed by the diagnostic task to determine $\lim_{x\to 1} \frac{x^2-1}{x-1}$. It showed a clear shift in meaning. Students' responses could be grouped into three categories. The first group correctly stated that the meaning of limit is that as x approaches 1, then $f(x)$ approaches 2, this happens even though $f(x)$ is not defined when x =1, using the idea of "approaching." The second group focused on factorization and substitution as methods for obtaining the limit value. The third group reduced the concept of limit to direct substitution of x into the function.

Discussion

The analysis of didactical transposition reveals shifts in the meaning of limit from scholarly knowledge to learned knowledge (Chevallard, 2019). In scholarly knowledge, limit is expressed through the rigorous structure as a relational and quantified concept. Through external didactical transposition, this formal meaning is introduced in the textbook in a more intuitive form, emphasizing the idea of "approaching a value" (Varberg et al., 2007). Although this simplification makes the concept more accessible, it also reduces some of its epistemological depth. During internal didactical transposition, the lecturer further transformed the concept by emphasizing algebraic procedures such as substitution, factorization, and cancellation. Consequently, students' learned knowledge reflected intuitive and procedural meanings more strongly than the formal structure of limit.

These findings are consistent with Tall and Vinner's (1981) distinction between concept image and concept definition. The students' concept images were mainly shaped by intuitive descriptions and operational techniques, while the formal definition was largely absent from their explanations. Similar tendencies have been identified by Cornu (1991) and Tall (2004), who showed that students often interpret limits dynamically or procedurally when instruction emphasizes calculation rather than conceptual meaning. In this study, however, this divergence is understood not only as a cognitive difficulty but also as a consequence of how the concept was transformed across textbook presentation and classroom instruction.

The three response patterns illustrate this transformation. The first pattern interpreted limit as a value approached by the function even when the function was undefined at the point, reflecting a dynamic conception of limit (Cornu, 1991). The second emphasized factorization and substitution, illustrating the proceduralization of mathematical concepts described by Sfard (1991) and Artigue (2000). The third reduced limit to direct substitution, consistent with findings that students may equate a limit with evaluating the function at a point (Nardi, 2008; Tall & Katz, 2014). These responses suggest that the intuitive meaning introduced in the textbook and the procedural emphasis in classroom instruction were reflected in students' understanding.

The contribution of this study therefore lies not in identifying entirely new learning obstacles, but in tracing how they emerge through the interaction between external and internal didactical transposition. Previous studies have emphasized that students' opportunities to develop conceptual understanding are influenced by how mathematical knowledge is selected, organized, and institutionalized in teaching (Schmidt & Winsløw, 2021; Topphol, 2023; Pocalana & Robutti, 2025). By connecting scholarly knowledge, knowledge to be taught, taught knowledge, and learned knowledge with students' actual responses, this study provides a concrete explanation of how excessive simplification and procedural emphasis may create an epistemological distance between the formal meaning of limit and students' understanding.

CONCLUSION

This study shows that the meaning of the limit function undergoes a systematic transformation through both external and internal didactical transposition. At the level of external transposition, the formal ε–δ definition is simplified into intuitive and procedural forms in textbooks, narrowing the epistemological scope of the concept. At the level of internal transposition, this simplification is further reinforced in classroom practice through a focus on algebraic manipulation and substitution. As a result, students' learned knowledge is dominated by operational and intuitive interpretations, as reflected in the three identified response patterns, indicating a clear divergence from the formal mathematical meaning of limit. These findings confirm that students' learning obstacles are produced through the combined effects of external and internal didactical transposition.

This study is limited by the small number of participants from a single institution and by its focus on the intuitive definition of limit without an in-depth exploration of students' engagement with the formal ε–δ definition. Future research should involve a broader range of institutions and participants, and incorporate multiple types of tasks, including graphical, numerical, and formal problems. Further studies are also needed to move beyond diagnosis by developing and testing didactical designs or learning trajectories that explicitly bridge intuitive and formal meanings of limit, supporting students in constructing a coherent and conceptually grounded understanding.

REFERENCES

Adhikari, K. P. (2021). Difficulties and misconceptions of students in learning limit. Interdisciplinary Research in Education, 5(1–2), 15–26. https://doi.org/10.3126/ire.v5i1-2.34731

Artigue, M. (2000). Teaching and learning calculus: What can be learned from education research and curricular changes in France? In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), Research in collegiate mathematics education IV (pp. 1–15). American Mathematical Society. https://doi.org/10.1090/cbmath/008/01

Artigue, M. (2014). Didactic engineering in mathematics education. In S. Lerman (Ed.), Encyclopedia of mathematics education (pp. 159–162). Springer. https://doi.org/10.1007/978-94-007-4978-8_44

Barbé, J., Bosch, M., Espinoza, L., & Gascón, J. (2005). Didactic restrictions on the teacher's practice: The case of limits of functions in Spanish high schools. Educational Studies in Mathematics, 59(1–3), 235–268. https://doi.org/10.1007/s10649-005-5889-z

Baye, M. G., Ayele, M. A., & Wondimuneh, T. E. (2021). Implementing GeoGebra integrated with multi-teaching approaches guided by the APOS theory to enhance students' conceptual understanding of limit in Ethiopian universities. Heliyon, 7(5), Article e07012. https://doi.org/10.1016/j.heliyon.2021.e07012

Brousseau, G. (1997). Theory of didactical situations in mathematics: Didactique des mathématiques, 1970–1990 (N. Balacheff, M. Cooper, R. Sutherland, & V. Warfield, Eds. & Trans.). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47211-2

Chevallard, Y. (1991). La transposition didactique: Du savoir savant au savoir enseigné (2nd ed.). La Pensée Sauvage. https://revue-rdm.com/ouvrage/la-transposition-didactique/

Chevallard, Y. (2019). Introducing the anthropological theory of the didactic: An attempt at a principled approach. Hiroshima Journal of Mathematics Education, 12, 71–114. https://doi.org/10.24529/hjme.1205

Chevallard, Y., & Bosch, M. (2014). Didactic transposition in mathematics education. In S. Lerman (Ed.), Encyclopedia of mathematics education (pp. 170–174). Springer. https://doi.org/10.1007/978-94-007-4978-8_48

Chevallard, Y., & Bosch, M. (2020). Didactic transposition in mathematics education. In S. Lerman (Ed.), Encyclopedia of mathematics education (2nd ed., pp. 214–218). Springer. https://doi.org/10.1007/978-3-030-15789-0_48

Cornu, B. (1991). Limits. In D. Tall (Ed.), Advanced mathematical thinking (pp. 153–166). Kluwer Academic Publishers. https://doi.org/10.1007/0-306-47203-1_10

Cottrill, J., Dubinsky, E., Nichols, D., Schwingendorf, K., Thomas, K., & Vidakovic, D. (1996). Understanding the limit concept: Beginning with a coordinated process scheme. The Journal of Mathematical Behavior, 15(2), 167–192. https://doi.org/10.1016/S0732-3123(96)90015-2

Creswell, J. W., & Poth, C. N. (2018). Qualitative inquiry and research design: Choosing among five approaches (4th ed.). SAGE Publications. https://collegepublishing.sagepub.com/products/qualitative-inquiry-and-research-design-4-246896

Fischbein, E. (1987). Intuition in science and mathematics: An educational approach. D. Reidel Publishing. https://doi.org/10.1007/0-306-47237-6

Grabiner, J. V. (1983). Who gave you the epsilon? Cauchy and the origins of rigorous calculus. The American Mathematical Monthly, 90(3), 185–194. https://doi.org/10.1080/00029890.1983.11971185

Hong, D. S. (2023). Examining opportunities to learn limit in widely used calculus textbooks. International Journal of Science and Mathematics Education, 21(3), 881–898. https://doi.org/10.1007/s10763-022-10273-7

Katz, M. G., & Sherry, D. (2013). Leibniz's infinitesimals: Their fictionality, their modern implementations, and their foes from Berkeley to Russell and beyond. Erkenntnis, 78(3), 571–625. https://doi.org/10.1007/s10670-012-9370-y

Nardi, E. (2008). Amongst mathematicians: Teaching and learning mathematics at university level. Springer. https://doi.org/10.1007/978-0-387-37143-6

Özmantar, M. F., Akkoç, H., Bingölbali, E., Demir, S., & Ergene, B. (2010). Pre-service mathematics teachers' use of multiple representations in technology-rich environments. Eurasia Journal of Mathematics, Science and Technology Education, 6(1), 19–36. https://doi.org/10.12973/ejmste/75224

Pocalana, G., & Robutti, O. (2025). Mathematics teacher educators' documents, praxeologies, and beliefs: A holistic model. Journal of Mathematics Teacher Education, 28(2), 291–314. https://doi.org/10.1007/s10857-024-09656-0

Schmidt, K., & Winsløw, C. (2021). Authentic engineering problems in service mathematics assignments: Principles, processes and products from twenty years of task design. International Journal of Research in Undergraduate Mathematics Education, 7(2), 261–283. https://doi.org/10.1007/s40753-021-00133-0

Sebsibe, A. S., & Feza, N. N. (2020). Assessment of students' conceptual knowledge in limit of functions. International Electronic Journal of Mathematics Education, 15(2), Article em0574. https://doi.org/10.29333/iejme/6294

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. Educational Studies in Mathematics, 22(1), 1–36. https://doi.org/10.1007/BF00302715

Stewart, J. (2016). Calculus (8th ed.). Cengage Learning. https://www.cengage.com/c/calculus-8e-stewart/9781285740621/

Sulastri, R., Suryadi, D., Prabawanto, S., Cahya, E., & Fitriani, F. (2022). Zone of concept image differences in infinite limits at undergraduate level. Jurnal Didaktik Matematika, 9(1), 1–21. https://doi.org/10.24815/jdm.v9i1.24709

Tall, D. (1993). Students' difficulties in calculus. In Proceedings of Working Group 3 on Students' Difficulties in Calculus, ICME-7 (pp. 13–28). https://homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot1993k-calculus-wg3-icme.pdf

Tall, D. (2004). Thinking through three worlds of mathematics. In M. J. Høines and A. B. Fuglestad (Eds.), Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education (Vol. 4, pp. 281–288). International Group for the Psychology of Mathematics Education. https://eric.ed.gov/?id=ED489653

Tall, D., & Katz, M. G. (2014). A cognitive analysis of Cauchy's conceptions of function, continuity, limit, and infinitesimal, with implications for teaching the calculus. Educational Studies in Mathematics, 86(1), 97–124. https://doi.org/10.1007/s10649-014-9531-9

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. Educational Studies in Mathematics, 12(2), 151–169. https://doi.org/10.1007/BF00305619

Tonra, W. S., Suryadi, D., Mulyaning, E. C., & Kusnandi. (2025). Learning obstacles and the didactical design for teaching the limit of function in a calculus course. Journal on Mathematics Education, 16(1), 153–172. https://doi.org/10.22342/jme.v16i1.pp153-172

Topphol, V. (2023). The didactic transposition of the fundamental theorem of calculus. REDIMAT—Journal of Research in Mathematics Education, 12(2), 144–172. https://doi.org/10.17583/redimat.11982

Varberg, D., Purcell, E. J., & Rigdon, S. E. (2007). Calculus (9th ed.). Pearson Prentice Hall.

Viirman, O., Vivier, L., & Monaghan, J. (2022). The limit notion at three educational levels in three countries. International Journal of Research in Undergraduate Mathematics Education, 8(2), 222–244. https://doi.org/10.1007/s40753-022-00181-0

Williams, S. R. (2001). Predications of the limit concept: An application of repertory grids. Journal for Research in Mathematics Education, 32(4), 341–367. https://doi.org/10.2307/749699

Winsløw, C. (2007). Didactics of mathematics: An epistemological approach to mathematics education. The Curriculum Journal, 18(4), 523–536. https://doi.org/10.1080/09585170701687969

Winsløw, C. (2011). Anthropological theory of didactic phenomena: Some examples and principles of its use in the study of mathematics education. In M. Bosch, J. Gascón, A. Ruiz Olarría, M. Artaud, A. Bronner, Y. Chevallard, G. Cirade, C. Ladage, & M. Larguier (Eds.), Un panorama de la TAD (pp. 117–138). Centre de Recerca Matemàtica https://researchprofiles.ku.dk/en/publications/anthropological-theory-of-didactic-phenomena-some-principles-of-i/