Journal of Research and Advances in Mathematics Education

Volume 7, Issue 4, October 2022, pp. 253 – 272
DOI: 10.23917/jramathedu.v7i4.15303
p-ISSN: 2503-3697, e-ISSN: 2541-2590

JRAMathEdu

Student teachers' conceptualizations of mathematical problem solving and the nature of their warrants

Zakaria Ndemo [1]*, David K. J. Mtetwa[2]

[1]*Bindura University of Science Education, Zimbabwe*
[2]*University of Zimbabwe, Zimbabwe*

Citation: Ndemo, Z., & Mtetwa, D. (2022). Student teachers' conceptualizations of mathematical problem solving and the nature of their warrants. JRAMathEdu (Journal of Research and Advances in Mathematics Education), 7(4), 253-272. https://doi.org/10.23917/jramathedu.v7i4.15303

ARTICLE HISTORY:
Received 11 May 2022
Revised 1 August 2022
Accepted 5 September 2022
Published 30 October 2022

KEYWORDS:
Mathematical problem solving
Mathematical argumentation
Student conceptualizations
Mathematical warrants

ABSTRACT

Absence of inquiry about meaning of mathematical objects learners deal with has permeated the school mathematics curriculum. Deep learning through questioning situations can be achieved if learners are helped to embrace the argumentation discourse in their problem-solving efforts. This paper reports a study that investigated linkages between the kinds of mathematical arguments constructed by students and the students' grasp of the concept of problem solving. It aims to analyze the relationship of data obtained from questionnaires with Likert-style scales and textual data from 30 undergraduate students and 5 graduate students of mathematics education. The two data sources were triangulated with reflective interviews guided by students' written responses and their validations on likert items. Descriptive statistics were applied to Likert data and directed content analysis was used to analyze and interpret the qualitative data. The study concluded that students lacked appropriate conceptions of the notion of problem solving and in particular their thinking as reflected in their arguments that contradicted current understandings of the construct. The fragile grasp of the idea of mathematical problem solving uncovered by this study has the potential to inform mathematics instruction.

INTRODUCTION

Motivation and context for the study

There is a growing appreciation in mathematics education that mathematical problem solving should be central to all students' learning experiences because of the following reasons. *First*, problem solving is essential for deep learning as it helps to foster mathematical connections and multiple representations of concepts (Winata et al. 2020). The process of generating an argument for problem solving purposes involves seeking an explanation or justification of mathematical objects underlying the problem at hand, thereby promoting self-explanation and learning (Schwarz et al., 2010). In addition, arguments constructed during problem solving in order to refute or substantiate propositions underpinning the mathematical problem for which a solution is needed to deepens one's understanding of the problem space (Schwarz et al., 2010). Furthermore, mathematical problem solving that embraces the argumentation discourse allows the problem solver to interweave the premises, conclusions (solutions), limitations, and rebuttals , thereby improving the organization of mathematical knowledge (Schwarz et al., 2010, p. 121).

Second, the researchers argue that mathematics education students who will teach mathematics upon completion of studies should possess sturdy knowledge of the notion of problem solving. If the student teachers' knowledge of the concept of problem solving is lacking, they will teach the concept poorly. In other words, they are less likely to be able to promote among learners abilities to construct explanations of relationships shared by the mathematical objects embedded in

*Corresponding author: zndemo@gmail.com

a task for which a solution would be sought (Dede, 2019). Furthermore, Dede (2019) suggests that teachers should be able to foster justification skills among students which prompt learners' abilities to question situations. Rosyidi and Kohar (2018) write that justifying a mathematical proposition using valid arguments differentiates mathematics from other disciplines. Questioning situations is an essential process in conjecturing solutions during problem solving as suggested by Dede (2019). *Third*, student teachers' problem-solving efforts are likely to delineate the most significant ways in which higher order thinking and reasoning manifest. In other words, mathematical problem-solving lies at the heart of what mathematicians do and hence problem solving should be a focal idea in mathematics educators' quest to determine whether higher order thinking skills are fostered among student teachers. In addition, the National Council of Teachers for Mathematics (NCTM, 2000) recommends that teachers should be encouraged to "seek, formulate, and criticize explanations so that classes become communities of inquiry" (p. 346). Mathematical problem solving which is supposed to be an integral and essential aspect of the community of practice proposed by NCTM (2000) has faced many constraints. Hence, understanding student teachers' thinking around the notion of mathematical problem solving is an important step towards efforts intended to convert classrooms into communities of inquiry.

Fourth, problem solving involves argumentation which fosters conceptual change among problem solvers. According to Jonassen and Kim (2010), conceptual change occurs when students modify their understanding of concepts and the conceptual frameworks that encompass those concepts. Hence, the argumentation process is associated with meaning making as problem solvers learn through reflective interactions.

Despite the apparent ubiquity of problem solving in everyday life and at school, student teachers continue to exhibit severe difficulties with mathematical problem- solving (Mamona-Downs & Downs, 2013). Hence, despite benefits presented here that are associated with embracing argumentation in problem solving one of the difficulties plaguing students is that they do not have an appropriate conception of what constitutes a mathematical problem (Weber, 2001). Student teachers have been reported to argue with blinders in mathematics classrooms because they lack skills to foster argumentation. In other words, they are inadequately prepared to teach topics dependent on their abilities to construct mathematical arguments such as groups and abstract algebra (Akkurt & Durmus, 2022). Hence, they will not create opportunities for learners to engage in arguments as they would be inept at generating valid arguments. The argument is that if they lack a deep understanding of the concept of problem solving, then they are likely to systematically misapply it (Weber, 2001).

Further, simply recognizing or defining a mathematical idea does not guarantee that one can apply it appropriately in problem solving contexts. For instance, despite the fact that mathematical problem solving is a major topic in one of the undergraduate mathematics education professional courses titled: "*Pedagogical Issues in Mathematics Education*" in Teacher Education Curricula in Zimbabwe, many graduate teachers cannot distinguish between routine exercises and tasks that can be deemed authentic mathematical problems. For instance, it is common among mathematics teachers to hear such utterances as: "*I have given you ten problems to solve on Quadratic Equations,*" when as a matter of fact, they would be referring to routine tasks given to students as an exercise to reinforce concepts. The concepts may include the factorization method, method of solution by completing square, and solving by use of general quadratic formula. Hence, one of the major goals of this study is to investigate student teachers' conceptualization of the concept of mathematical problem solving. This study seeks to investigate students' understandings or thoughts of the notion of a mathematical problem and what the problem-solving process entails. Exploring student teachers' thoughts about linkages between problem solving is important because one's proficiency in the argumentation discourse is crucial in developing and communicating mathematical knowledge (Akkurt & Durmus, 2022).

Problem solving in mathematics has been characterized by impasses experienced by student teachers, a scenario whereby they simply do not know what to do in order to resolve mathematical task at hand. Mason et al. (1982) cited in Liljedahl, Santos, Malaspina and Bruder (2016) write that impasses experienced by students are an "honorable thing" as there is much to be learned from the condition of being stuck. Therefore, a question arises as to how accessible resources are to the students and what problem solving strategies are wielded by students during problem solving.

Although there is a growing body of literature on problem solving in mathematics, the ability of student teachers to generate arguments which illuminate their appreciation of the important factor of control remains unclear (Carlson & Bloom, 2005; Jonassen & Kim, 2010). Carlson & Bloom (2005) define control mechanisms as referring to problem solvers' abilities to access and implement relevant mathematical resources and strategies pertinent to a mathematical task at hand.

Briefly, control denotes metacognitive activities and decisions that influence the solution path. Describing the third phase of his problem solving model, Polya (1957) cited in Liljedahl et al. (2016) writes that we have to see how the various items are connected and how the conclusion is linked to data in order to obtain an idea of how to make a solution plan. We argue that the connections between the data and the solution(s) are illuminated by the kinds of warrants generated by the students. Hence, this study responds to the call for more investigations into the impact of control during problem solving. Another goal of this study is to develop an explanation about the kinds of warrants generated by students during their attempts to resolve given mathematical tasks and evaluate mathematical assertions about problem solving. The research paper were motivated by the desire to gain a better understanding of how the factor of control plays itself out during student teachers' problem solving attempts. The influence of control would be illuminated by the kinds of warrants formulated during problem solving. This study, which is informed by the idea of problem solving as some form of argumentation, sought to evaluate the sorts of evidence adduced by in-service mathematics education students as they engage with tasks assigned and describe the problem solving process.

Research questions

Accordingly, the research aimed first to determine the students' conceptualizations of the notions of mathematical problem solving, and second to establish the kinds of warrants generated by students during problem solving. The 2 questions posed are how do mathematics education students conceptualize the notion of mathematical problem solving and what kinds of warrants are generated by the students during problem solving?

Mathematical problem solving

In this section, the research paper explicate the general views adopted by the mathematics education community with regard to the notion of mathematical problem solving. A problem is defined by Mayer (2003) in Greiff et al. (2013) as any given state where a goal state needs to be reached and there is no routine method of solution available. Greiff et al. (2013) describe problem solving as the subsequent process of transforming the given goal state into the desired goal. Lovett (2002) notes that problem solving involves a phase of knowledge acquisition during which the problem solver tries to establish a representation of the problem. The knowledge acquisition process is then followed by the application process where there is a design of the solution plan and implementation of the solution plan to generate the solution (desired goal).

Mathematical problem solving involves students' engagement on any mathematical task that is not judged to be merely procedural, that is, a student does not have an initial overall idea about how to proceed with the process of generating a solution (Mamona-Downs & Downs, 2013). Hence, a problem will only a problem if you do not know how to go about solving it. In addition, Schoenfeld (1982) points out that a problem that has no surprises in store and can be solved comfortably by routine procedures no matter how difficult an exercise is.

The words no "surprises" captured in Schoenfeld's (1982) description of a mathematical problem is indicative of the fact that problem solving as a process involves situations in which a student is confronted with unfamiliar or non-routine tasks, as emphasized by Greiff, et al. (2013). Hence, the same point of novelty of the tasks the students engage with in problem solving is revealed in the phrase "not judged procedural" by Mamona-Downs and Downs (2013). The researchers observe that problem solving encompasses those situations for which a problem solver does not have access to the solution schema. In those situations his or her behavior is radically different from what we would see when a problem solver engages with a routine or familiar non-routine problem (Schoenfeld, 1982). A problem solver in those problem spaces (non-routine problem contexts) is no longer proficient and may not even have a clue about how to begin the problem-solving process

(Liljehadhl et al., 2016). A problem solver's efforts may even involve exploring the problem context and trying to match the problem with familiar problems. According to Liljehadhl et al. (2016), one may be tempted to bring a variety of plausible problem contexts like related facts, tentative approaches, and related problems. This description of the problem-solving process reveals a distinction in problem solving as an intentional act which occurs in a mechanical fashion versus problem solving which is neither intentional nor mechanical. The focus of this study is on the later approach to problem solving in mathematics.

Underpinnings of mathematical problem solving

For each individual there exist problems that will not yield to his or her intentional and mechanical attack. Literature on problem solving strategies (e.g., Greiger & Galbraith, 1998; Liljehadhl et al., 2016; Schoenfeld, 1982) suggest two main phases involved in solving such problems namely, entry and attack. Liljehadhl et al. (2016) explain that the entry phase involves the problem solver's efforts to develop familiarity with the problem at hand by examining special instances of the problem. Mason et al. (1982) describe the attack phase as a phase that involves conjecturing. Conjecturing involves formulating and testing hypotheses in an effort to develop a deep understanding of the problem and generate a solution. If attempts to generate the solution fail, the problem solver will revert to the entry phase. As the problem solver oscillates between the entry and attack phases, it is possible at some point to get stuck, acording to Mason et al. (1982), describe as being an "honorable and positive state from which much can be learned" (p. 55). Mason et al. (1982) further explain that the solver then reverts to the oscillation between entry and attack until an AHA is reached that enables him/her to generate a solution. This piece of literature on impasses and oscillations during problem solving is important to this study as it forms the lens to examine students' problem-solving behaviors as they engage with tasks or evaluate mathematical assertions about mathematical problem solving.

Studies on problem solving have reported that the relationship between the problem solver and the problem matters more than the perceived level of difficulty as viewed within some hierarchy of abstraction. Hence, the characteristics of the problem solver are significant in problem solving (Liljehadhl et al., 2016). The argument is that characteristics of the problem solver are revealed through the solver's conceptualizations of the problem-solving process as well as kinds of warrants the solver generates to link his/her solution to data.

Drawing from the large amount of literature related to problem solving, we examine ideas of resources and control that are relevant to the focus of the study. Carlson and Bloom (2005) define mathematical resources as formal and informal knowledge about the content domain including definitions, lemmas, theorems, and mathematical facts, procedural and conceptual knowledge. In addition, the word *resources*, when applied to problem solving, refers to key competences with respect to rules of the discourse. For example, a problem solver resolving a task that requires use of contrapositive needs to recognize that the nature of the mathematical discourse is such a problem solver should begin by negating the consequence of the implication statement $p \Rightarrow q$. The task is then accomplished through some direct deduction that should lead to the conclusion that the negation of p is false. Carlson & Bloom (2005) noted that the utility of resources depends on the level of accessibility of those resources to the problem solver. So, the word *control* is used to address the question: how accessible are the relevant resources? while an individual may possess resources, he/she may fail to access those resources in the context of producing a solution. Commenting on the influence of control as a factor Schoenfeld (1992) cited in Liljehadhl et al. (2016) says "It is not just a matter of what we know; it's how, when, and whether you use it" (p. 48). Problem solving tendencies associated with poor control mechanisms include random associations, absence of inquiry about meaning, and lack of validation efforts.

Random associations include situations in which strategies and algorithms are used with blinders, for example, students' tendency to solve the quadratic equation when they are supposed to factorize the expression $f(x) = ax^2 + bx + c$. Absence of inquiry refers to cases whereby one does not reflect on the essence of solutions generated. For example, solutions such as $x = -3$ should be discarded when solving equations involving $logx$. Closely related to the factor of control is the notion of self-regulation which refers to actions taken by a problem solver in response to one's assessments of problem solving efforts (Carlson & Bloom, 2005). It can be inferred from the discussion that control consists of metacognitive behaviors and decisions that influence the problem-solving path. Metacognitive behavior refers to actions and decisions that reflect a problem solver's awareness and monitoring of an individual's thought processes. These pertain to the selection of resources and implementation of problem solving techniques.

Cognitive engagement during problem solving involves a problem solver's efforts to establish connections between the problem scenario and his/her prior knowledge. Cognitive engagement during problem solving also includes efforts to construct logically connected propositions (warrants) that underline solution of the problem. One of the main objectives of the current study is to determine the kinds of warrants generated by student teachers. It can be noted from the discussion of the problem-solving process that problem is the essence of doing mathematics (Sen & Guler, 2015)

Toulmin's argumentation model

An overview of the model is given by Schwarz, Hershkowitz, and Prusak (2010) who write that Toulmin's argument layout model is a conclusion to be drawn from data supported by a warrant that might be justified by an endorsement. Jonassen and Kim (2010) write that Toulmin developed an argumentation structure comprising the elements: data (D), claim or conclusion (C), warrant (W), backing (B), qualifier (Q) and rebuttal (R). In line with Toulmin's argumentation scheme, *first,* we have to make a claim (C) by asserting something (Ubuz, Dincer & Bübül, 2013). For our challenger who poses a question such as, "What have you got to get there?" The facts an arguer appeals to as a foundation for the claim made are called data (D). After providing data our challenger can inject into the argumentation process another question such as, "How did you get there?" The purpose of posing such a question is to force the arguer to demonstrate that his/her step from data to the claim was an appropriate one by giving different kinds of propositions in the form of rules, principles, and inferences that are called warrants (Ubuz et. al., 2013). Ubuz, et al. (2013) observe that warrants are of different kinds and confer with different degrees of force on the conclusions they justify. Ubuz et al. note that an arguer may then decide to put in a qualifier (Q) such as "necessarily," "probably," "for every $\varepsilon > 0$" to the degree of force to which data confer on the conclusion in virtue of the warrants. However, in some cases, it is possible to envision exceptional conditions which are capable of undermining the claim thereby leading to its refutation. These conditions capable of injecting contradiction in the argumentation process are called rebuttals (R). Yet, another question such as, "Why do you think that?" can be posed by our challenger concerning the acceptability of our warrants. Ubuz, et al. (2013) call our answer to this question the backing (B).

An argument serves the purpose of justifying a claim on the basis of data (Ubuz et. al., 2013). In this study, researchers use the word argumentation as a process of weighting evidence for or against a solution to a problem. Kuhn (1991) noted that providing evidence in support of claims is an important skill. However, Walton (1996) reports that most arguers often provide inadequate evidence in their efforts to substantiate assertions made. The argumentation process has been characterized by the following flaws. *First*, the most common weakness in the argumentation discourse is that people have a propensity to provide supporting evidence as opposed to generating disconfirming evidence (Maxwell & Mittapalli, 2010). In other words, arguers have shown a tendency to select evidence that only support their claims which Perkins et al.(1991) refer to as "my-side-bias," a greater conviction to personal beliefs than counter evidence. Other deficiencies in the argumentation discourse shown by most arguers include overgeneralizing shown by inferring from a few cases or a single case considered (Weber & Mejia-Ramos, 2015).

METHODS

Research design

Ideas discussed in the Theoretical Background informed the major research design decisions for this study including instrument development. For example, the definition of a mathematical problem described in the Theoretical Background helped the researchers to ensure that novel tasks constituted the task- based interviews. The purpose of the study is to understand student teachers' thoughts about the concept of solving mathematical problems, develop an understanding of the types of guarantees they produce to justify their solutions, and influence researchers to use case study designs (Creswell, 2014). Furthermore, the data analysis plan was formed by the constructs in the

Theoretical Background. For instance, Ubuz et. al. (2013) description of the notion of a warrant in Toulmin's (2003) model of argumentation served as a lens for discerning the kinds of facts students appealed to in their efforts to justify mathematical claims made as they engaged with the tasks.

Participants

The study involved 35 participants: 30 undergraduate and 5 postgraduate mathematics education students at one state university in Zimbabwe. The undergraduate student teachers were in semester one of the second year of study. The undergraduate student teachers had studied two mathematics education courses during the first year of their studies: *Pedagogical Issues in Mathematics* and *History and Philosophy of Mathematics*. History and Philosophy of Mathematics course includes the topic; *Problem Solving in Mathematics* in which aspects of problem solving are treated largely from a theoretical perspective. During the first-year semester two theoretical ideas about problem solving are then applied in lesson planning and peer teaching sessions slotted for the course: *Pedagogical Issues in Mathematics*. Hence, it was anticipated *Pedagogical Issues in Mathematics* that student teachers would provide rich data about problem solving in the sense suggested by Charmaz (2014).

The undergraduate student teachers were holders of a Diploma in Education that had prepared them to teach mathematics at the secondary school level. The Diploma in Education for secondary school mathematics teachers offered through Teachers' Colleges affiliated to the Department of Teacher Education (DTE) of the University of Zimbabwe (UZ) covers three broad subject areas, including pure mathematics, mechanics, and statistics. For the pure mathematics section, pre-service teachers study content pitched to Advanced level. Topics in the pure mathematics section include Functions, Series and Sequences, Integration, Differentiation, and Vectors. With regards to the mechanics section, the DTE curriculum for mathematics majors covers topics such as Forces and Equilibrium Conditions of particles, Equations of motion of a particle, Newton's Laws of motion, Work, Energy and Power among other Advanced Level mathematics subject matter. Topics in the Statistics section of the DTE curriculum for secondary mathematics teachers include Probability, Discrete Random Variables, and Continuous Random Variables. Key concepts such as expectation, and variance are examined in the statistics component of the DTE curriculum. We mention that for both Mechanics and Statistics sections concepts covered in the Pure Mathematics section are applied to solve problems. The task in section C of the research instrument was drawn from the Mechanics section of the curriculum just described. In other words, the undergraduate student teachers involved in the current study had enrolled for continuing professional development (CPD). The CPD programme for mathematics include content courses in mechanics (for details of the CPD programme structure for the undergraduate student teacher informants see Ndemo & Mtetwa, 2021).

The study also involved 5 Master of Science Education Degree students majoring in mathematics education at the same university. Being postgraduate student teachers, they had studied the same or similar courses at undergraduate level and so the researcher reasonably assumed that rich data could be elicited from the postgraduate students. Further, the post graduate students had on average taught mathematics at high school level for at least 5 years. In closing this section, we mention that the rationale for including postgraduates in the current study was to get a sense of the prevalence or intensity of mathematics teachers' struggles with generating arguments during problem solving. We had no intention of making some comparisons between postgraduates and undergraduates. We argue that getting a sense of how the problem had permeated the curriculum could in turn inform CPD and other curriculum decisions. For instance, those responsible for content driven CPD initiatives would be in a position to decide whether workshops on problem-solving activities should include master degree holders.

Data collection procedure

Data were generated in two phases. Phase one involved administering the questionnaire with three sections. After doing some preliminary data analysis, we selected five undergraduate student teacher informants whom we engaged with in reflective interviews guided by their responses to the questionnaire.

Data collection: Task-based interview

Data collection took place during week 12 of the 15 week long semester for both undergraduate and postgraduate study participants. A sheet of paper with three sections of our research instrument was given to each student teacher. The questionnaire had spaces provided for writing answers. Student teachers worked through the items independently. There were no time restrictions imposed during data collection. The instrument had three sections A, B, and C. The researchers elaborate on the data collection procedure by attempting to articulate how theoretical constructs from the Theoretical Background influenced the data collection procedure.

For section A, Likert data were elicited. The section consisted of assertions made about the concept of problem solving. The theoretical positions discussed under Theoretical Background informed our formulations. For instance, theoretical underpinnings of the concept of problem solving such as the notion of novelty, a problem being non-routine, non-availability of solution schema, characteristics of problem solver (Lovett, 1982; Greiff et al. 2013; Mamon-Downs & Downs, 2013; Schoenfeld, 1992), influenced the manner in which we formulated the likert items. Participants have validated propositions made about a mathematical problem and the problem-solving process (Selden & Selden, 2003). This technique of generating likert data is consistent with the process of measuring a latent variable by Punch (2005). Punch suggests that once a variable has been defined, a researcher should proceed to produce indicators for the variable in order to measure it. Some items from Section A of the task-based interview are now reproduced:

A1: A mathematical problem involves difficult tasks.
Strongly Agree Agree Disagree Strongly Disagree
A3: A good problem should be solved by intentional and procedural means.
Strongly Agree Agree Disagree Strongly Disagree

Informants were instructed to encircle their responses. The participants' responses to items in section A informed questions included in the reflective interview guide. For instance, the student would be asked to explain why she agreed with the assertion *Mathematical problem solving involves difficult tasks*. There were ten Likert items. In Section B, participants described mathematical problem solving and went on to describe activities that characterize problem solving. The intent was to explore their thoughts on the idea of a mathematical problem. Moore (1994) in Maya and Surmarmo (2011) noted that a student can write the definition of a concept without understanding it fully. Following Styliandes and Stylianides (2009), the student teachers' validations of mathematical assertions concerning problem solving were followed by a phase where the student teachers constructed the definition of a mathematical problem and described activities involved in problem solving. In other words, the combined construction-evaluation strategy suggested by Stylianides and Stylianides (2009) was considered strategic for this study. Furthermore, to determine whether participants had a deep grasp of problem solving, it was then considered strategic to engage with some task that constituted Section C of our research instrument. In Section C participants engaged with the task:

Solve the following task: A force of $2\sqrt{2}N$ acts along the diagonal AC of a square ABCD and another force P acts along AD. If the resultant force is inclined at $60°$ to AB, find the value of P.

The task in section C was drawn from high school mathematics subject content. Hence, it was deemed to be within the informants' conceptual reach because they had covered the concepts during initial teacher training when they studied for the Diploma in Education. Student teachers took about 45 minutes to complete the three sections of the research instrument.

Data collection: Interviewing

As stated in the introductory remarks to this section, the researchers scrutinized the student teachers' responses and developed an interview guide on the basis of informants' reasoning as they engaged with the tasks. For instance, one of the 5 undergraduate student teachers was asked to justify why he agreed that mathematical problem solving involves difficult tasks. During this phase of data collection student, teachers engaged in reflective discourse with the interviewer with respect to their solution attempts and responses to likert items (Bleiler, et al., 2014). Interviewing took place in the mathematics lecture room during mid-morning and it took about 30 minutes to interview a respondent. The interview exchange was recorded of course with the consent of the participants, using a timed audio recorder.

Researchers considered that the verbal formulations would illuminate the kinds of thoughts held about problem solving as well the kinds of warrants used to link C and D elements. Follow- up interviews were informed by Dahlberg and Housman's (1997) notion of a learning event. A learning event is said to have occurred when an individual communicates his/her understanding of a concept. A communication could be in the form of an utterance or in written form as participants justified the basis of their arguments when solving the task assigned.

Consistent with Toulmin's model, we drew from van Eemeren et al. (1996) who propose that the researchers' position with respect to the task assigned should not be taken into consideration but we allowed the participants to write or describe their warrants and construct their definitions of a mathematical problem without interference. From Toulmin's model we focused on the nature of connections of three components: data (D), claim (C) and warrant (W) as these provided a window through which we could view student teachers' thoughts about problem solving.

Further, the concept of argumentation was also seen as a verbal and social activity of reasoning aimed at increasing (decreasing) the acceptability of a controversial standpoint by producing a constellation of propositions intended to justify or refute a mathematical claim (Ubuz, et al., 2013). This perspective of argumentation was found to be strategic during the interviewing as we sought clarifications on how they had evaluated assertions made with respect to the concept of mathematical problem solving. The perspective of argumentation as a verbal and social activity also applied to instances when we sought clarity on written responses produced during informants' problem-solving efforts. Hence, the constructs discussed in the Theoretical Background shaped the data collection instruments and the subsequent data analysis strategies used.

Reliability and validity issues

In this section, we examine potential threats to credibility of the study and measures employed to reduce their effects. *First,* we observed that it was difficult to record accurately student teachers' conceptualizations of problem solving because not all of it is conscious (Selden & Selden, 2003, p. 5). Clearly, it is not possible to verbalize non-conscious activity through 'think aloud' verbal protocols. Consequently, it is possible that student teachers could fail to report crucial components of their arguments (Inglis & Alock, 2012). Further, asking students to verbalize their thinking might reduce the level of cognitive resources informants devoted to reflective practice about their problem-solving efforts. Hence, generated data may not reflect with total accuracy student teachers' thoughts about problem solving and the kinds of their warrants as they engaged with the task. However, asking participants to verbalize their thinking might actually increase their efficacy at arguing as suggested by Inglis and Alcock (2012). Hence, the researchers were confident that data collected could be used to adequately address research questions.

Second, interview transcripts were checked to ensure they do not contain mistakes. During transcribing the audio recorders were played several times. *Third,* another validity measure used is prolonged interaction (Creswell, 2009; Maxwell & Mittapali, 2010). The study took place during a 15 week long semester during which the first author played the researcher-tutor dual role. Prolonged time with informants enabled the researcher to develop an in-depth understanding of their conceptions of problem solving (Creswell, 2009). Third, triangulation was employed. Data from the three sources were compared for consistency with respect to teacher informants' thoughts about the concept of problem solving as well as the kinds of propositions, rules or principles articulated in an effort to connect the C and D elements. Finally, discussions held with student teacher informants during follow up interviews to their problem solving attempts served as some of member checks (Creswell, 2014; Maxwell & Mittapali, 2010). Hence, seeking clarity to student teachers' responses helped to clear misinterpretations we had about informants' problem-solving attempts. For instance, member checking led to realization that despite the fact that participants possessed the relevant pieces of knowledge, they could not access these knowledge elements because of poor control mechanisms suggested by Carlson and Bloom (2005)

With regards to reliability concerns of the study, the researchers employed the split-half method of assessing reliability of data proposed by Lewis (2009). The split-half method measures reliability of data by formulating several questions to evaluate the same idea. Hence, we leveraged on vast literature on problem solving to construct items examining the same idea about problem solving. For example, in section A the items:

A3: Problem solving involves tasks for which one is not aware of the method of solving
A10: Problem solving involves routine and familiar mathematical tasks,

were intended to evaluate the students' level of grasp of the notion of novelty in problem solving. Similarly, items:

A2: Problem solving is about engaging with tasks for which we know the relevant formulas and steps.
A4: A good problem should not be solved by intentional and procedural means,

were meant to assess the student teacher informants' conceptions of the crucial idea that there is no routine method of moving from the present state to the desired goal state (Greiff et al., 2013). Thus, although the items were formulated differently, they explored the same idea about problem solving. Hence, responses from these items were checked for consistency (reliability) depending on the student teachers' comprehension of problem solving.

Data analysis

Data analysis was realized in three phases. First, the researchers first analyzed likert data from Section A by exploring descriptive statistics measures, bearing in mind that it was ordinal and hence statistical measures such as mean and standard deviation were invalid for the likert data (Puri, 1996). Frequencies were applied to the ordinal data to get a sense of the student teachers' conceptions of the notion of problem solving.

Next, researchers applied directed content analysis to student teachers' written efforts. Hsieh and Shannon (2005) define content analysis as a careful, detailed and systematic examination and interpretation of textual data in order to identify patterns, themes, biases and meanings. According to Hsieh and Shannon, researchers can apply conventional (summative) content analysis technique or directed content analysis technique. Conventional content analysis is used to develop codes and categories that emerge inductively from data (Berg, 2009, p. 339). In other words, with one begins with actual words and phrases to develop codes and themes. Directed content analysis allows the researcher to immerse herself/himself in data guided by codes and categories derived from existing theories and those that can possibly emerge from data. Codes and categories derived from existing theories are called analytic codes.

Hence, following Sen and Guler (2015), the researchers examined student teachers' written attempts through the lenses of vast ideas drawn from the background. For instance, Toulmin's model of argumentation was used to discern the kinds of rules and principles they appealed to in order to justify the connections between C and D elements. In addition, literature ideas about problem solving behaviours namely: oscillating between entry and attack phases, the notions of impasses and types of arguments informed directed content analysis of written responses to the task. Directed content analysis technique was considered to be appropriate because it allowed us to stay open for inductive codes in those instances in which there was no one to one mapping of student teachers' responses with existing underlying ideas about argumentation for the purpose of problem solving.

Finally, we examined the interview transcriptions of follow up interviews to check for consistency between claims made during problem solving efforts and nature of conceptualizations held about the problem-solving process. For each participant data consisted of: a student teacher' responses to likert items, written attempt to the assigned task and where considered necessary an interview transcript of follow up interview. Two sets of data were produced for the 35 student teacher informants. Together, researchers then applied data analysis procedure just described to two data sets for 10 undergraduate student teachers. Following Sen and Guler (2015), researchers then analyzed the remaining data from 20 undergraduate and 5 post graduate student teachers independently and then met during agreed intervals to compare analysis reports. In those instances where results differed, researchers discussed their data analysis reports until some consensus was reached. As concluding remarks to this section, we note that in-vivo codes were used to support inferences drawn from informants' responses (Corbin & Strauss, 2008).

FINDINGS AND DISCUSSION

Research question one: *How do the student teachers conceptualize the concept of mathematical problem solving?* Content analysis technique was applied to qualitative data elicited from section B of the questionnaire in which participants described the problem-solving process. The following categories emerged from our data.

In Table 1, it can be seen that the undergraduate student teachers conceived mathematical problem solving as involving using formulas presumably to do calculations as implied by the phrase "plugging into the formula where" an answer is to be found. This category emerged as dominant with (12 out of 31) responses revealing such thoughts. Therefore, the leading belief (Furighetti & Morselli, 2011) among the participants was that mathematical problem solving is about using formulas to resolve tasks. Further, the students expressed the sentiments that the relevant formulas are provided by the teachers or from sources such as text books. Responses such as the one by Tapiwa:

> **Tapiwa**: A task for which we know the relevant formulas [...],

reveal a fragile understanding of the concept of a mathematical problem since Tapiwa held the view that formulas should be known prior to engaging with tasks. Such thoughts are consistent with results from analysis of likert data from section A item A2 where students evaluated the assertion:

> **A2**: Problem solving is about engaging with tasks for which we know the relevant formulas and steps.

Item analysis revealed that student teachers had a weak command of the idea of novelty as a key requirement for the problem-solving process as 80% (24 out of 31) responses agreed with the assertion.

Contrary to current thinking about the discourse of problem- solving the informants expressed that problem solving involves difficult or challenging tasks. From Table 1, it can be seen that 16% (5 out of 31) responses were in this vein. This is a contradiction to current thinking about problem solving in mathematics education where focus is now on the relationship between the problem solver and the problem at hand. Greiger and Galbraith (1998) reported that it is the relationship between the problem and the solver that matters and not the level of difficulty of the problem as seen within some hierarchy of abstraction. This inference was also supported by frequencies from likert data where 80% (24 out 30) of the participants accepted the claim that:

> **A1**: A mathematical problem involves difficult tasks.

And yet it is the characteristic of the problem solver that is a more important consideration in problem solving (Carlson & Bloom, 2005; Liljhedhl et al., 2016).

In Table 1, it can be seen that another leading belief held by the student teachers in connection with the concept of problem solving was that the problem solving process requires a problem solver to engage in sequential steps and procedures to obtain a solution with 23% (7 out 31) of the responses in this categories. A more revealing picture about this category was illuminated by frequencies from likert data from section A item A4:

> **A4**: A good problem should be solved by intentional and procedural means.

Item analysis revealed that 87% (26 out of 30) agreed with the assertion that problem solving involves resolving tasks using intentional and mechanical means—a leading belief that is in sharp contrast to current thoughts about problem solving in which tasks that count as problems should not yield to routine and mechanical methods (Carlson & Bloom, 2005; Liljehadhl et al., 2016). Participants' fragile understanding of problem solving was further revealed by item analysis of item A3:

> **A3**: Problem solving involves tasks for which one is not aware of the method of solving,

as 73% (22 out of 30) disagreed with this statement.

Overall, the study has revealed that students had a weak command of the concept of problem solving. In addition, there was only (1 out of 31 responses in the category in which a problem was conceptualized in terms of justifying a mathematical claim see Table 1. Further, Table 1 shows that only (1out of 31 responses) belonged to the category in which problem solving involves analyzing. These observations revealed student teacher informants' struggles with higher order cognitive skills as captured by the words 'analyzing' and 'justifying' used to describe the inductive categories. These findings are a cause for serious concern for the teacher education curriculum given that the study involved participants who had enrolled for in-service degree studies. In other words, the informants had studied the area of problem solving during their initiation into the teaching profession at diploma level.

Thus far pseudonyms were used to report results from undergraduate student teachers. To differentiate postgraduate students' results from those of undergraduate group, the coding PGS was used. PGS stands for postgraduate student. So PGS1 and PGS2 were used to refer postgraduate student teacher one and student teacher two respectively. From the postgraduate group the dominant thought was that problem is about using formulas to generate a solution. Examples of such thoughts include the following:

PGS2: […] a problem which needs to be calculated using a mathematical formula to get an answer.

PGS5: […] stating the formula and applying the formula […] procedural approach.

These responses reveal that the postgraduate students' ideas around the concept of problem solving were similar to those expressed by undergraduates who described problem solving as involving "plugging into the formula" a term captured here by word 'calculation' from the undergraduate group and also by the word 'figure' by PGS4. The conception of problem solving as a process involving use of formulas was mentioned by (3 out of 5) mathematics education postgraduate students. Directed content analysis of data elicited from the postgraduates revealed that problem solving involves tasks that PGS2 said "can be solved by step by step (procedural) methods." The crucial element of novelty was mentioned by:

PGS5: The solver to the problem is not familiar with the problem and seeks to find answers to the problem.

Hence, the words "is not familiar" revealed that the postgraduate mathematics education student had a good grasp of the idea that problem solvers should engage with non-routine tasks. Data analysis revealed that the informants appreciated the importance of previous knowledge as was echoed in the following responses:

PGS4: […] the problem solver to apply previous knowledge.

PGS5: requires strong background knowledge to match with current problem.

Analysis of likert data revealed that all 5 post graduate student teachers disagreed with the assertion that problem solvers should be exposed to relevant methods and formulas before engaging with tasks. Hence, while responses by PGS4 and PGS5 illustrate the importance of previous knowledge in problem solving, the fact that the same postgraduates appreciated the need not to expose learners to methods first indicates that the student teachers had a deeper grasp of the element of novelty during problem solving. However, these findings are not consistent with observation that (3 out of 5) responses agreed with the statement that a good mathematical problem can be resolved by intentional and mechanical means. Hence, the informants lacked an appropriate conception of problem solving because there was lack of consistency across items. Finally, similar to findings from the undergraduates the master's degree student teachers also agreed with the statement that problem solving involves difficult tasks—a result that conflicts with current thoughts that place emphasis on the relationship between a problem and a person seeking to resolve it.

Table 1: Categories of undergraduate students' conceptualizations of the mathematical problem-solving process

Category	Example	Frequency $n = 30$
Using a formula	Thoko: Plugging into the formula where an answer is needed.	12
Requires steps and procedures	Crispen: Requires steps and procedures so as to come with a solution	5
Applying learnt concepts	Mica: Mathematical problem solving requires to apply learnt concepts	7
Challenging or difficult tasks	Kuda: These are tasks which considered to be difficult	5
Justifying truth	Raina: [...] to justify if the statement is true	1
Analyzing using mathematical methods	Hilda: These are problems which can be solved, analyzed using mathematical methods	1
	Total	31

Research question two: *What kinds of warrants are generated by the student teachers during problem solving?*

To address research question two content analysis technique was applied to the textual data that consisted of student teachers' written responses to the task in section C of the task-based interview. Ideas drawn from Toulmin's (2003) model of argumentation as well as from vast literature on problem solving provided some lens for examining the kinds of rules, principles, and inferences (warrants) employed by students to justify their claims as they engaged with a mathematical task assigned. Furthermore, reflective interviews were held in order to obtain a revealing picture on the sorts of reasoning displayed by the participants as they engaged with the task (Ndemo & Mtetwa, 2017).

Researchers now examine Crispen's effort to solve the theoretical mechanics task in Figure 1. it can be seen that Crispen made the inference that the diagonal AC is inclined at 60° to the horizontal instead of being inclined at 45°. Further, it can also be inferred from Figure 4 that one of the given forces, $2\sqrt{2}$ N was represented as the resultant which is another flawed reasoning exhibited by Crispen. These two wrong inferences (warrants) marred Crispen's solution attempt despite the fact that he could correctly apply the trigonometric ratio, $sin\theta = \frac{opposite}{hypotenuse}$ to the right angled triangleADC to get $P = \sqrt{2} \times \sqrt{3} = \sqrt{6}$. In other words, Crispen could not infer that the forces P and $2\sqrt{2}$have a resultant inclined at 60° to the horizontal AB. Crispen's weak command of underlying ideas pertinent to the task became apparently clear when he obtained another value $P = \sqrt{2}$. He did not reflect on the problem solving process to get its essence. In other words, two different values of P ($P = \sqrt{2}$and $P = \sqrt{6}$) were not questioned. This shows absence of inquiry about meaning (Mason, Burton & Stacey, 1982).

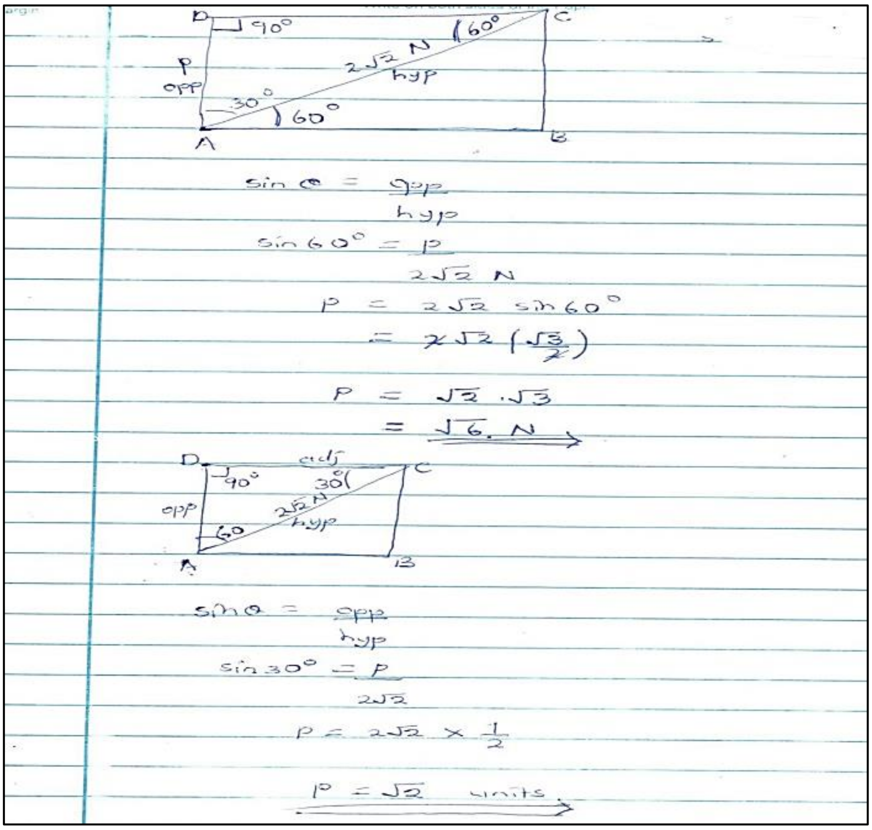

Figure 1: Crispen's problem solving effort

It can, therefore, be inferred from Crispen's effort that there was no conscious evaluation of 'tools' drawn from his repertoire (Greiff et al. 2013). Lack of conscious evaluation resulted in Crispen's failure to represent the problem (Lovett, 2012). Hence, he could not even reach the entry phase of problem solving (Mason et al. 1982). It can be concluded that Crispen was just roaming outside the problem space. Further, it can be noted that his C and D elements were disjointed. Researchers observed that one's ability to construct mathematical arguments during problem solving is a key indicator of his/her mathematical competence as suggested by Kartika et al. (2021). The researcher probed Crispen to get a sense of the kinds of inferences he made during the problem-solving process. An extract of the exchange that took place between the first author and Crispen is now reproduced.

Researcher	**: [**...] At Form 1 and Form 2 you examined properties of a square. What does the diagonal do the interior angles?
Crispen	**:** *Inomaita ma* 45°. (It bisects the angles).
Researcher	**:** But you labeled here the angle as 60° [referring to the diagonal *C* in Figure 1.
Crispen	**:** It was about the resultant force.
Researcher	: What about it?
Crispen	: It was inclined at 60°.
Researcher	**:** Do you have anything more to say?
Crispen	**:** *Zvakwana* (I no longer have anything to say).

Just as was in Thoko's case while Crispen was aware that in a square the diagonal bisects the angles by saying "*inomaita ma* 45°,he could access this resource that could have allowed him to resolve the $2\sqrt{2}$N force horizontally and vertically to determine the components of the resultant. Finally, Crispen could have employed the concept, $\tan\alpha = \frac{Y}{X}$, where Y and X denote the vertical and horizontal components respectively of the resultant force and α is the angle the resultant makes with the horizontal axis. Hence, poor control mechanisms caused Crispen to engage with the task with

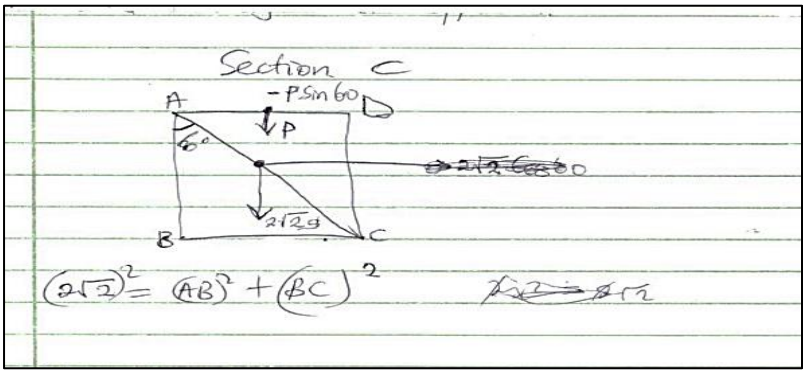

Figure 2: Felix's written response to task

blinders which forced him to create a 'special diagonal' of a square inclined at 60° instead of 45° (Carlson & Bloom, 2005). Crispen's problem solving behavior was typical of those with poor control mechanisms shown by failure to access the relevant resource from his repertoire despite the fact that he possessed it (Schoenfeld, 1992). Similar to Thoko, Crispen failed to negotiate the entry phase (Mason et al., 1982). He also roamed outside the problem space making claims that showed that C and D elements were disjointed. However, it is encouraging to observe when Crispen criticized his reasoning during the follow up interview managed to develop an understanding of the underlying ideas by pointing out that '' It was about the resultant force…. It was inclined at 60°. So Crispen was involved in critical thinking that led to the realization of the error he had made (Graham & Lesseig, 2018) Next, we discuss Felix's problem solving behavior.

Figure 2 shows that Felix could not even negotiate the entry phase of the problem-solving process (Schoenfeld, 1982) since he showed evidence of not understanding the problem. While the task stipulates that force P acts along AD,Figure 2 illustrates that P is almost perpendicular to AD. Similarly, while it was stated in the task that the $2\sqrt{2}N$force acts along the diagonal AC of the square $ABCD$, Figure 2 illustrates that the force acts vertically with its initial point of application being the midpoint of the of the diagonal AC. Furthermore, the statement indicates that the $2\sqrt{2}N$force now represents the diagonal of the square $ABCD$, an assertion that is contrary to Felix's illustration (inference) that the force acts vertically. Felix's problem solving behaviour is a typical example of what Carlson and Bloom (2005) refer to as random associations that are indicators of poor control mechanisms during problem solving. Put differently, the idea is that Felix could not access relevant resources needed to resolve the task. It was startling that Felix could Part of the interview transcription gives a clearer picture of the problem-solving behavior exhibited by Felix.

Researcher : […] Why did you write it as 60°?
Felix : Eee, if the resultant force is inclined at 60° to AB. […] so my diagonal […]. I treated the diagonal as the resultant force.
Researcher : What is one of the properties of the diagonal of a square?
Felix **:** Diagonal of a square, eee, it divides, eee, because this angle is 90°, so it becomes 45°.

The interview excerpt reveals that Felix possessed the resource (in a square the diagonal bisects angle) but he could not access it in the context of producing a solution of the task at hand (Carlson & Bloom, 2005). Furthermore, other relevant resources like the concept of resolving a vector, method of determining the direction using the relation $tan\theta = \frac{Y}{X}$ were also not accessed. In other words, Felix's repertoire of relevant prior knowledge was seriously depleted to the extent of keeping him outside the problem space. These findings from the undergraduate mathematics education student teachers are consistent with Mason, Burton and Stacey's (1982) description of a mathematical problem. Mason et al. argue that when engaging with real authentic problem, problem solving proficiency drops drastically even for experts. Furthermore, they argue that impasses experienced during problem solving can be so severe that an expert may not even know how to start. Hence, while the study revealed serious disconnections between C and D elements the findings

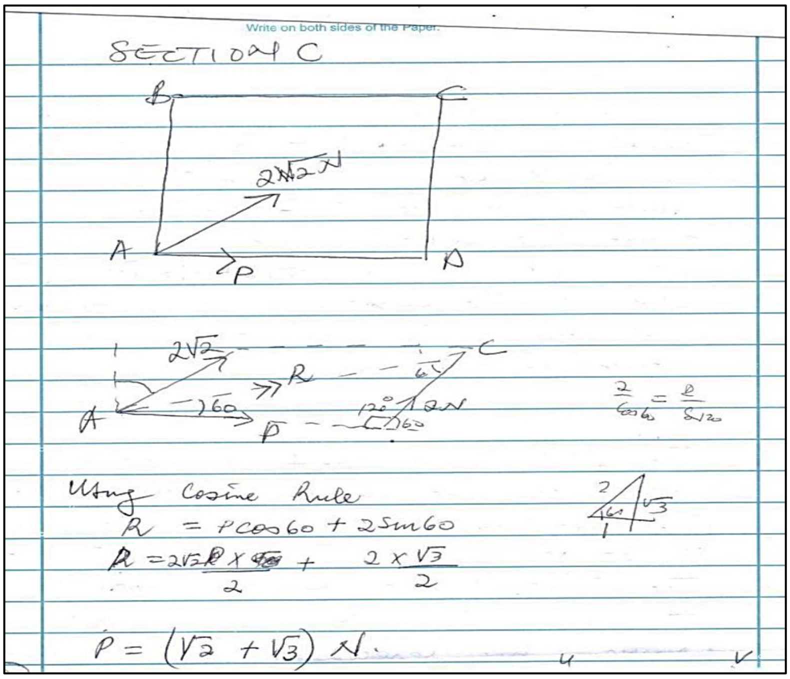

Figure 3: PGS4's written response to Theoretical Mechanics task

confirm and reinforce the idea that there is indeed a distinction between problem solving and intentional and mechanical attacks of routine tasks.

Next, we present results from content analysis of postgraduate students' written efforts on the same task. From Figure 3, it can be observed that the resultant R is inclined at 60° to AD instead of AB as stipulated by the item. From the same figure, it can be seen that triangle ACD was considered to be an isosceles triangle because it is $C\hat{A}D = A\hat{C}D = 60°$. Further, quadrilateral ABCD was then illustrated as a parallelogram. However, the opposite sides have different sizes of $2\sqrt{2}N$ for AB and $2N$ for DC yet in a parallelogram opposite sides are parallel and equal. Once again, such claims reveal that PGS4 did not evaluate the knowledge elements appealed to and hence he could not develop a representation of the problem (Mason et al., 1982; Lovett, 2002). Next, PGS4 then wrote the claim: "Using the Cosine Rule" which was then followed by the statement: "$R = Pcos60° + 2sin60°$." We observe here that the statement: R=Pcos60°+2sin60°, has no logical connection with the Cosine Rule as asserted by PGS4. Furthermore, the statement: $R = P\cos 60^0 + 2\sin 60^0$ is false because from the triangle of vectors we are supposed to have $\overrightarrow{AC} = \overrightarrow{AD} + \overrightarrow{DC}$ that would lead to $\vec{R} = \vec{P} + 2$. The statement: "R=P cos 60°+ 2 sin 60°" was then followed by: "$R = \frac{2\sqrt{2}}{2}P + \frac{2\sqrt{3}}{2}$", from which we can deduce that PGS4 inferred that $\cos 60^0 = \sqrt{2}$ –a wrong statement since $|cos\theta| \leq 1$. Finally, the quantity P vanished from the expression: $R = \frac{2\sqrt{2}}{2}P + \frac{2\sqrt{3}}{2}$ and the answer: $R = \sqrt{2} + \sqrt{3}$ just sprang from nowhere. Such actions and decisions by PGS4 reveal poor self-regulation that prompted the student teacher to reason with blinders.

PGS4's problem solving behavior illustrates that he had a shaky grasp of elementary Theoretical Mechanics concepts because being a postgraduate student it was expected that the concepts examined were within his conceptual reach. This is so because postgraduate students are holders of a bachelor degree in mathematics education that capacitate them to teach Theoretical Mechanics concepts at high school level. Furthermore, wrong inferences (warrants) made about the Cosine Rule and wrong inferences about properties of a parallelogram illustrate the importance of building a coherent structure in one's previous knowledge needed to solve a task. Hence, wrong inferences were generated because of lack of coherence in the PGS4's pieces of relevant prior knowledge. As was in many other cases PGS4 roamed outside the problem space. Such impasses are typical of the problem-solving endeavor (Mason et al., 1982). There was no conscious evaluation of

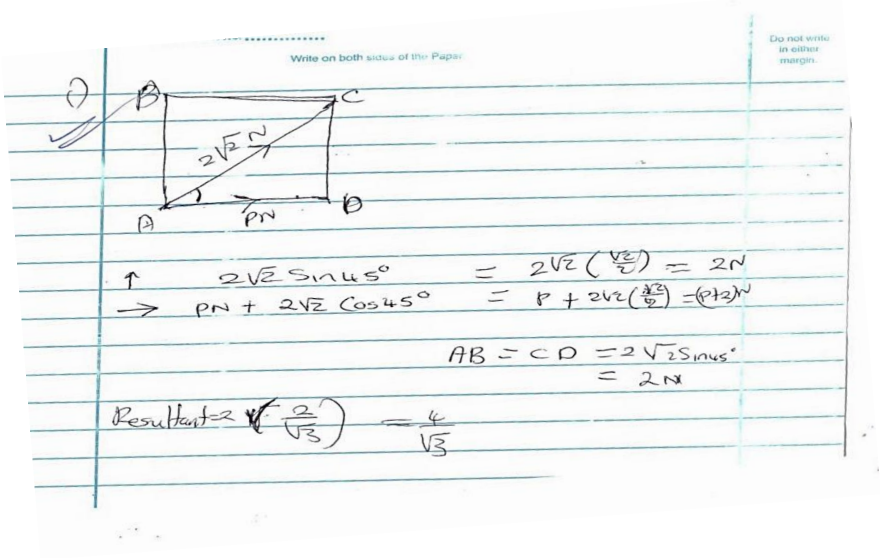

Figure 4: PGS2's written response to Theoretical Mechanics task

claims made (Greiff et al., 2013). For instance, it is worrisome that a postgraduate failed to realize that $\cos 60^0 = \sqrt{2}$ does not hold. In other words, we infer that the claims (C) made are disjointed from the data (D) produced.

Finally, we examine PGS2's written response to the Theoretical Mechanics task. Figure 4 illustrates that PGS2 made the correct inference that the diagonal AC of square $ABCD$ bisects the right angle BAD. From the same Figure 4, it can also be seen that PGS2 could access the resource: resolution of forces that enabled him to determine the vertical and horizontal components of the resultant force as shown in Figure 4. Components of the resultant force were determined to be $2N$ and $P + 2$ respectively for the vertical component and horizontal component as indicated by arrows on Figure 4.

PGS2 could negotiate the entry phase of the problem as demonstrated by his understanding of the concept of resolving a force system. In other words, relevant pieces of knowledge were extracted from PGS2's repertoire namely: diagonal of square bisects angles, notion of resolution of forces (Greiff et al., 2013). Not only was the student teacher able to represent the problem, PGS2 produced correct symbols for vertical and horizontal components of the resultant (Lovett, 2002). Hence, it can be inferred that there was a good linkage between C and D elements. In other words, claims made by PGS2 were supported by relevant mathematical formulations. However, it was during the attack phase of problem-solving process that the student experienced some impasse and he wrote: "resultant $2\left(\frac{2}{\sqrt{3}}\right) = \frac{4}{\sqrt{3}}$." PGS2 could not access the resource that the direction of the resultant is given by $\tan\theta = \frac{Y}{X}$, where Y denotes the vertical component of the resultant and X refers to horizontal component of the resultant force. Hence, he could not make progress with the solution seeking exercise. PGS2's problem solving effort illustrates that sometimes a problem solver can lack access to relevant resources, a condition that can then cripple one's problem solving activity.

CONCLUSIONS

From the discussion of results, the following conclusions can be drawn. For research question one, *first,* student teachers conceptualized problem solving as a process that involves use of formulas in sequential steps. Such thoughts indicate a weak command of the problem-solving process since the participants expressed the sentiments that the formulas and steps should be known prior to engaging with tasks. For instance, 73% (22 out of 30 held the belief that one should be aware of the method of solving. Conceiving problem solving in this manner revealed a shaky grasp among the participants of the crucial concept of novelty in problem solving whereby a person seeking to resolve a problematic task need not be in possession of the solution schema. This is because possessing a solution schema renders that task to a routine exercise no matter its level of difficulty. In addition to engaging in steps and procedures that are supposed to be known before encountering problems, the student teachers also thought that such steps and procedures should be provided by teachers. Hence, participants thought of problem solving as a process one cannot accomplish alone without the assistance of someone considered knowledgeable in the area.

Second, students involved in this study conceived problem solving as a process of engaging with difficult mathematical tasks. Likert data from undergraduate group revealed that 80% (24out of 30) and all post graduate student teachers agreed with the assertion that mathematical problem solving is about engaging with difficult tasks. This finding points towards the fact that the informants had not yet embraced current thinking in mathematical problem-solving discourse, which has seen a shift from the level of difficult of problem to an emphasis on the relationship between the problem and a person seeking to resolve it. So overall, the study has revealed that the sample of undergraduates and postgraduates lack an appropriate conception of the notion of problem solving.

With respect to research question two, the following conclusions were made. Participants demonstrated poor control mechanism. Despite possessing resources needed to solve the task at hand they could not utilize those resources in the context of producing a solution (refer to Thoko's solution attempt in the results section). Poor control mechanisms exhibited by both postgraduate and undergraduate student informants resulted in a scenario whereby the students engaged in the problem-solving activities with blinders. Consequently, random associations were a pronounced feature of the students' problem-solving efforts (refer to Felix's and PGS2's solution attempts). In other words, the kinds of warrants generated by informants were disconnected with the data. In terms of Toulmin's model the study revealed gaps between data (D) and claims (C) made during student teachers' problem-solving attempts. These findings could be explained in terms of the notion of random associations because was lack of inquiry about meaning. For instance, Crispen did not question the values $P = \sqrt{2}$ and $P = \sqrt{6}$ for the same task where a single answer was required. In addition, the study has uncovered that problem solvers sometimes experience severe difficulties during problem solving that can force them to violate established mathematical rules and truths. In other words, the respondents generated irrelevant warrants. For example, some participants introduced a 'special diagonal' of a square that divided a right angle into $30°$ and $60°$ angles instead of bisecting the right angle.

Overall, the study has revealed that the student teachers had unhelpful conception of problem solving with the dominant thought that the process involves difficult tasks. For instance, student teachers justified the need to know formulas well ahead of the problem-solving process to avoid what Crispen called 'waffling', that is, going astray. Hence, mathematics educators should strive to foster stronger conceptions of mathematical problem solving. In addition, educators and mathematicians should address deficiencies uncovered by the study. First, although problem solving is a process in which higher order thinking should manifest there were few responses from the sample that illuminated high cognitive skills. In conclusion, the authors of this paper note that these are important ideas that can inform mathematics instruction.

Contribution and looking ahead

The current study has uncovered an inappropriate representation of the notion of problem solving in which the assertions/claims (C) and data (D) were disjointed as they were not supported by accurate warrants (W) (e.g., the cases of Thoko, Felix, PGS2 and PGS4). This is a worrisome finding given that the gaps in informants' reasoning manifested not only at undergraduate level but even among postgraduates. It will lead to concepts examined would be within their conceptual reach because the task involved was drawn from high school subject content they are supposed to teach comfortably to their own students. The disconnections between D and C were so pronounced that student teachers could not even negotiate the entry phase and hence they kept roaming outside the problem space (the case of Crispen).

Overall, the student teacher's thinking expressed by this study contradicts current thinking in mathematics education discourse where the relationship between problem and problem solver has become prominent in front of task difficulty. Next, we came to the conclusion regarding the fragile understanding of problem solving among participants who reasoned that problem solving should involve sequential steps and procedures. In addition, the random associations and lack of inquiry about meaning exhibited by informants point to poor control mechanisms during problem solving. Indeed, as anticipated the fragile understanding of the concept of a problem was accompanied by non-robust linkages between C and D elements. Briefly, we argue that the disconnections between the D and C elements of Toulmin's model and the non-convincing conceptualizations of problem solving can count as new ideas uncovered by this study. Hence, this study has contributed to efforts meant to develop an understanding of how informants conceptualize the notion of problem solving on the basis of their actual productions or voices as opposed to validating arguments supplied by researchers.

In our investigation we excluded other elements of Toulmin's model namely, the qualifier, Q, the rebuttal, R, and the backing, B as we thought that doing so would render this piece clumsy. Literature (Maxwell and Mittapali, 2007, 2010; Perkins, 1991) has reported that arguers are good at providing supporting evidence but so inept at producing disconfirming evidence, the so called 'my-side bias'. Hence, counter-argumentation skill is crucial in pedagogical issues in mathematics education. Therefore, we suggest that further research involving the element of rebuttal, *R* is vital. Furthermore, we propose further studies into ways of narrowing the undesirable disconnections between D and C elements.

ACKNOWLEDGMENT

The researchers express their appreciation of comments made by peers during conference and seminar presentations of this paper. Suggestions and comments made helped very significantly in improving the quality of the manuscript. Researchers also thank the informants for their patience during data collection.

BIBLIOGRAPHY

Akkurt,Y.Y., & Durmus, R. (2002). Tracing proof schemes: some patterns and new perspectives. *Journal of Research and Advances in Mathematics Education*, 7(1), 1-16.

Bleiler, S.K.,Thompson, D.R., & Krajčevski, M. (2014). Providing feedback on students'mathematical arguments: Validations of prospective secondary mathematics teachers. *Journal of Mathematics Teacher Education,* 17, 105-127.

Carlson, M.P., Bloom, I. (2005). The cyclic nature of problem solving: an emergent multidimensional problem-solving-framework. *Educational Studies in Mathematics*, 58, 58-75. https://doi.org/10.1007/s10649-005-0808-x

Charmaz, K. (2014). *Constructing grounded theory: A practical guide through qualitative analysis*. London: Sage.

Corbin, J., & Strauss, A. (2008). *Basics of qualitative research: Techniques and procedures for developing grounded theory* (3rd ed.). Sage Publications, Inc. https://doi.org/10.4135/9781452230153

Creswell, J.W. (2014). *Research design: Qualitative, quantitative and mixed methods approaches.* London: Sage.

Dahlberg, R. P., Housman, D. L. (1997). Facilitating Learning Events through Example Generation. *Educational Studies in Mathematics*, 33(3), 283-99. https://doi.org/10.1023/A:1002999415887

Dede, A.T. (2019). Arguments constructed within the mathematical modelling cycle. *International Journal of Mathematics Education in Science and Technology*, 50(2), 292-314.

Driver, R., Newton, P., & Osborne, J. (2000). Establishingthe norms of scientific argumentation in classrooms. *Science Education*, 84, 287-312. https://doi.org/10.1002/(SICI)1098-237X(200005)84:3%3C287::AID-SCE1%3E3.0.CO;2A

Furinghetti, F., & Morselli, F. (2011). Beliefs and beyond: hows and whys in the teaching of proof. *Mathematics Education,* 43, 587-599. https://doi.org/10.1007/s11858-011-0316-7

Graham, M., & Lesseig, K. (2018). Black-spot strategies for argumentation. *The Mathematics Teacher*, 112(3), 172-178. https://doi.org/10.5951/mathteacher.112.3.0172

Greiff, S.,Holt, V., & Funke, J. (2013). Perspective on problem solving in education: Analytical, interactive and collaboration problem solving. *The Journal of Problem Solving*, 5(2), 71-91.

Greiger, V., & Galbraith, P. (1998). Developing a diagnostic framework for evaluating students' approaches to applied mathematics. *International Journal of Mathematics Education, Science and Technology*, 29, 533-559.

Housman, D. & Porter, M. (2003). Proof schemes and learning strategies of above-average mathematics students. *Educational Studies in Mathematics,* 53, 139-158.

Jonassen, D.H, & Kim, B. (2010). Arguing to learn and learning to argue: Design justifications and guidelines. *Educational Technology Research and Development,* 58 (4), 439-457. https://doi.org/10.1007/s11423-009-9143-8

Kuhn, D. (1991). *The skills argument*. Cambridge: Cambridge University Press. https://doi.org/10.1017/CBO9780511571350

Kartika, H., Budiarto, M., & Fuad, Y. (2021). Argumentation in K-12 mathematics and science education: A content analysis of articles. *International Journal of Research on Education*, 7(1), 51-64. https://doi.org/10.46328/ijres.1389

Lewis, J. (2009). Redefining qualitative methods: Believability of the fifth moment. *International Journal of Qualitative Methods,* 8(2), 1-13. https://doi.org/10.1177/160940690900800201

Lawson, T. (2009). Cambridge sociology: an interview with Tony Lawson. *Erasmus Journal of Philosophy and Economics,* 2(1), 100-122. https://doi.org/10.23941/ejpe.v2i1.26

Liljedahl, P. , Santos-Trigo, M., Malaspina, U., & Bruder, R. (2016). *Problem solving in mathematics education. ICME-13 Topical surveys.* Hambury: Hamburg University. https://doi.org/10.1007/978-3-319-40730-2

Lovett, M.C. (2002). Memory and cognitive processes. In D. Medin (Ed), *Stevens handbook of experimental psychology*, 317-362.

Mamona-Downs, J., & Downs, M. (2013). Problem solving and its elements in forming proof. *The Mathematics Enthusiast,* 1(1), 136-160. https://doi.org/10.54870/1551-3440.1263

Mason, J., Burton, L., & Stacey, K. (1982). *Thinking mathematically*. Harlow: Pearson Prentice Hall.

Maxwell, J.A., & Mittapalli, K. (2010). Realism as a stance for mixed methods research. In A. Tashakkori and T. Teddlie (Eds), *Sage handbook of mixed methods in social behavioural research* (pp.145-166). Washington DC: Sage Publications. https://doi.org/10.4135/9781506335193.n6

Maya, R., & Sumarmo, U. (2011). Mathematical understanding and proving abilities: Experiment with undergraduate student by using modified Moore learning approach. *IndoMS, Journal of Mathematics Education,* 2(2), 231-250.

Moore, R. C. (1994). Making the transition to formal proof*. Educational Studies in Mathematics,* 27, 249-266*.* https://doi.org/10.1007/BF01273731

Perkins, D.N., Farady, M., & Bushey, B. (1991). Everyday reasoning and the roots of intelligence. In J.F. Voss, D.N.Perkins, & J.W. Segal (Eds.), *Informal reasoning and education* (pp. 83-106), Hillsdale, NJ: Erlbaum.

Punch, K.F.(2005). *Introduction to social science research: Quantitative and qualitative approaches.* London: Sage.

Puri, B.K. (1996). *Statistics in practice: An illustrated guide to SPSS.* New York: Oxford University Press.

Rosyidi, R., & Kohar, A.W. (2018). Student teachers' proof schemes on proof tasks involving inequality:deductive or inductive? *Journal of Physics Conference Series.*

Sen, C., & Guler, G. (2015). Examination of secondary school seventh graders'proof skills and proof schemes. *Universal Journal of Educational Research*, 3(9), 617-631. https://doi.org/10.13189/ujer.2015.030906

Schoenfeld, A.H.(1992). Learning to think mathematically: Problem solving, metacognitionand sense-making in mathematics. In D.A. Grouws (Ed.), *Handbook for Research on Mathematics Teaching and Learning,* Macmillan Publishing Company, New York, pp.334-370.

Schoenfeld, A.H. (1982). Some thoughts on problem solving research and mathematics education. In F.K. Lester & J. Garofalo (Eds.), *Mathematical problem solving: Issues in Research*. Franklin Institute Press, Philademia, pp. 27-37.

Schwarz, B. B., Hershkowitz, R. & Prusak, N. (2010). Argumentation and mathematics. In Littleton, K & Howe, C (Eds.), *Educational Dialogues: Understanding and Promoting Productive Interaction* (p. 103-127). London, UK: Routledge.

Selden, A., & Selden, J. (2003). Validations of proofs considered as texts: Can undergraduates tell whether an argument proves a theorem? *Journal for Research in Mathematics Education,* 34, 4–36.

Stylianides, A.J., & Stylianides, G.J. (2009). Proof constructions and evaluations*. Educational Studies in Mathematics,* 72(3), 237-253. https://doi.org/10.1007/s10649-009-9191-3

Toulmin, S. E. (2003). *The uses of argument*. Cambridge, UK: Cambridge University Press. https://doi.org/10.1017/CBO9780511840005

Ubuz, B., Dincer, S., & Büb*ül*, A. (2013). Argumentation in undergraduate math courses: A study on definition construction. In A.M. Lindmeier, & A. Heinze (Eds.), *Proceedings of the 37th Conference of International Group for the Psychology of Mathematics Education*: Vol. 4. (pp. 313-320). Kiel University: PME.

Van Eemeren, F., Grootendorst, R., & Henkemans, F.S. (1996). *Fundamentals of argumentation theory: A handbook of historical backgrounds and contemporary developments*. Mahwah. NJ: Erlbaum.

Varghese, T. (2009). Secondary-level teachers' conceptions of mathematical proof. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1, 1-13.

Vinner, S. (1997). The pseudo-conceptual and pseudo-analytical thought processes in mathematics learning. *Educational Studies in Mathematics*, 34, 97-129.

Winata, R., Friantini, R.N., Armurwanda, C., Annur, M.F., & Permata, J.I. (2020). The arguments of mathematics education students to solve proof problems. *AIPP conference Proceedings*.

Walton, D.N. (1996). Argumentation schemes for presumptive reasoning. Mahwah NJ: Erlbaum Associates.

Weber, K. (2001). Student difficulty in constructing proofs. The need for strategic knowledge. *Educational Studies in Mathematics,* 48, 101-119.

Weber, K., & Mejia-Ramos, J.P. (2015). On relative and absolute conviction in mathematics. *For the Learning of Mathematics*, 35(2), 3-18.