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Visualization of finite groups: the case of the Rubik's cube and supporting properties in GeoGebra

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ABSTRACT

In this text, we present a discussion about the Rubik's Cube and its relationship with Group Theory, particularly permutation groups, as well as possibilities for exploring it using the GeoGebra software interface. We bring a brief discussion about the concept of group, aspects of the Rubik's cube, the Rubik's group as a group of permutations and possibilities for its exploration in GeoGebra. Based on this study, we recognize the potential to delve into permutation groups in Abstract Algebra through a visual interface that associates their properties with a tangible and manipulable object. Additionally, there is the potential for simulating their movements using Dynamic Geometry software, such as GeoGebra. These findings highlight the relevance of GeoGebra as a useful tool for visualizing and understanding permutation groups, promoting a more intuitive and accessible approach to Group Theory in the educational context.

INTRODUCTION

In 1974, Ernö Rubik, a professor from Budapest, Hungary, introduced his fascinating invention called the Rubik's Cube. This mathematical toy became a part of popular culture, inspiring competitions, and captivating both children and geniuses who enjoyed the mental challenge of solving it (Carter, 2009). The problem posed by this game-like toy involves starting from a position where the faces display their smaller squares in different colors and performing a sequence of movements on the cube's axes so that all six faces show a single color (Joyner, 1996; 2008; Chen, 2004; Romero, 2013).

It is common to observe that students, even those enrolled in undergraduate mathematics courses, may face difficulties when attempting to solve the Rubik's Cube problem in a practical and objective manner. Additionally, there may be a lack of clear understanding regarding the mathematical properties involved in the necessary movements. According to Vágová and Kmetová (2018, p. 1054), "the importance of using mental images and visual processing in mathematics has been increasingly recognized." The authors also note that since the early '90s, the debate about the real existence of a mental image in the human brain has taken on a new form, especially with the availability of neuroimaging techniques.

This work explores aspects of permutation groups that underlie the construction of the Rubik's Cube from a visual perspective. In this way, we explore possibilities of manipulation and understanding of its structure with the support of the GeoGebra software. The aim of this work is to explore how visualizing permutation groups through GeoGebra can facilitate the understanding of abstract concepts of Group Theory, especially in educational contexts.

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GeoGebra is a powerful tool for illustrating mathematical concepts (Alves, 2019; 2022), including the relationship between the Rubik's Cube and permutation groups, providing a visual and interactive representation of the mathematical concepts involved.

The exploration was carried out using a practical approach with GeoGebra software, focused on visualizing the permutation groups associated with the Rubik's Cube. The analysis involved the application of specific permutation algorithms and the manipulation of the cube configurations within the software. The study did not involve a specific sample of participants, but relied on demonstrations and practical explorations to illustrate the theoretical concepts.

This work is an initial section of ongoing doctoral research, which seeks different ways of approaching Abstract Algebra and the algebraic structures of finite groups outside of traditional molds, in an attempt to explore visualization and increase the assimilation of their concepts.

Before discussing some aspects of the Rubik's Cube construction, let's briefly revisit the concept of a group in Abstract Algebra.

A brief description of the group concept

A non-empty set G, equipped with an operation $*$, is termed a group if, and only if, for every pair of elements (a, b) in G, a unique $c \in G$ can be associated, where $c = a * b$ (Gonçalves, 1995; Wussing, 1984; Davvaz, 2021). Equivalently, it is understood that there exists a function:

$$*: G \times G \to G$$
$$(a, b) \longmapsto a *$$

and for the pair G,* to be considered a group, the following properties must be satisfied:

(i) (Associative) $a * (b * c) = (a * b) * c$, for any $a, b, c \in G$.

(ii) (Existence of identity element) There exists $e \in G$, such that $a * e = e * a = a$, for any $a \in G$.

(iii) (Existence of inverses) For each $a \in G$, there exists $b \in G$ such that $a * b = b * a = e$. Additionally, if the property:

(iv) $a * b = b * a, \forall\, a, b \in G$ (commutative) is identified and satisfied, we say that the group is *abelian*.

In the case of finite groups, the number of elements in G is called the order of the group, and the operation table $*$ is called the group table. It allows us to analyze all possibilities and arrangements among its elements.

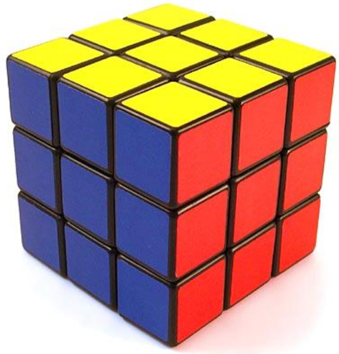
Aspects of Rubik's cube

For the understanding of the mathematics involved in the Rubik's Cube, we rely on some important works over the years on the subject, such as Warusfel (1981), Bandelow (1982), Chen (2004), Joyner (1997; 2008), Carter (2009), and Romero (2013). Here, we explore the most classic case of the cube, which is its 3x3 structure.

The Rubik's Cube has a different color for each of its faces. The colors of the cube's faces may vary depending on its place of manufacture. However, it is known that there are six distinct colors, and the most common colors are blue, red, yellow, green, white, and orange. The pieces of the Rubik's Cube (the small colored cubes) can be permuted in various ways through possible movements. Each configuration of the Rubik's Cube can be represented as a specific permutation of the pieces, where the goal when solving the cube is to reach a specific configuration, such as having all faces with a single color (Warusfel, 1981). We have an example of the cube in Figure 1.

However, it is worth noting that we do not focus on color for understanding its movements, but rather on its position. We can use the initial letters of the following terms to refer to the faces of the cube and the possibilities of movements as in Figure 1. To better understand the dynamics of each of the smaller facets, Joyner (1997, p. 69) provides us with a diagram (Figure 2).

This diagram allows us to visualize the layout of the 3x3 Rubik's Cube model, highlighting the orientation of each face as F, B, L, R, U, D on the central small cubes, as shown in Table 1. Thus, it is possible to check the generators corresponding to the six faces of the cube (Joyner, 1997), which, in turn, can be written in disjoint cycle notation as organized in Table 1.



Right = R	Front = F	Up = U
Left = L	Back = B	Down = D

Figure 1. Rubik's cube and movements

Table 1
Orientation on the central small cubes

F	(17, 19, 24, 22) (18, 21, 23, 20) (6, 25, 43, 16) (7, 28, 42, 13) (8, 30, 41,11)
B	(33, 35, 40, 38) (34, 37, 39, 36) (3, 9, 46, 32) (2, 12, 47, 29) (1, 14, 48, 27).
L	(9, 11, 16, 14) (10, 13, 15, 12)(1, 17, 41, 40) (4, 20, 44, 37) (6, 22, 46, 35)
R	(25, 27, 32, 30) (26, 29, 31, 28) (3, 38, 43, 19) (5, 36, 45, 21) (8, 33, 48, 24)
U	(1, 3, 8, 6) (2, 5, 7, 4) (9, 33, 25, 17) (10, 34, 26, 18) (11, 35, 27, 19)
D	(41, 43, 48, 46) (42, 45, 47, 44) (14, 22, 30, 38) (15, 23, 31, 39) (16, 24, 32, 40)

The mechanism allows each face the freedom of rotation from 0° to 360° around the axis that fixes the central small cube to the internal mechanism, both clockwise and counterclockwise, always taking the face in front of the manipulator as the reference. Each rotation of the cube makes a 90° movement, and by performing the same movement four times, completing a total of 360°, the cube returns to the initial position, making it a neutral operation (Joyner, 2008; Carter, 2009).

Note that when we rotate a face, the color of the central square on the face always shows the same color. Therefore, we identify each face by the color of its center. Thus, we use the six letters to designate the six faces, as well as various pieces and positions. For example, the four central parts of the edges corresponding to the U face will be UR, UF, UL, and UB, while the four parts of the vertices corresponding to the U face will be URF, UFL, ULB, and UBR. Note that UR and RU are the same piece. The colors of the vertices are ordered clockwise. Thus, URF, RFU, and FUR denote the same part.

The names of the faces are also used to refer to quarter-turn movements clockwise, where the face is oriented towards the manipulator. An example of this is: R involves a 90° clockwise rotation of the face to the right. The half-turn of the R face, whether clockwise or counterclockwise, does not matter; we denote it as R^2 because it corresponds to two 90° clockwise rotations. The counterclockwise rotation can be denoted as R^{-1} or R' for the sake of mathematical convention.

A sequence of movements is written from left to right. For example, RU means that the R movement is applied first, followed by U. The context allows us to distinguish whether a sequence of two or three letters corresponds to a sequence of movements or to a piece.

Rubik's group

As seen earlier, we designate as R, L, F, B, U, and D the clockwise movements of the right, left, front, back, up, and down faces, respectively. According to Chen (2004), the set of possible movements of the Rubik's Cube can be transformed into a group, which we can denote as $(G,*)$. Two

```
             +--------------+
             |  1    2    3 |
             |  4    u    5 |
             |  6    7    8 |
+------------+--------------+--------------+--------------+
|  9   10  11 | 17   18   19 | 25   26   27 | 33   34   35 |
| 12    l  13 | 20    f   21 | 28    r   29 | 36    b   37 |
| 14   15  16 | 22   23   24 | 30   31   32 | 38   39   40 |
+------------+--------------+--------------+--------------+
             | 41   42   43 |
             | 44    d   45 |
             | 46   47   48 |
             +--------------+
```

Figure 2. Diagram of each of the larger and smaller faces of a 3x3 Rubik's Cube

movements can be considered equal if they result in the same configuration for the cube; for example, rotating a face 180° clockwise would be the same as rotating 180° counterclockwise.

Thus, the group operation can be defined as follows: if M_1 and M_2 are two movements, then $M_1 * M_2$ is the movement where you first execute movement M_1 and then M_2. We call the set of all allowed movements in shuffling the cube the Rubik's Group or Group R. In this way, let's verify the existence of the three conditions that satisfy the group definition for the Rubik's Cube:

1st condition: Associativity. When we perform any sequence, for example, of 3 generic movements X, Y, Z, we observe that: $X(Y\,Z) = (XY)Z$. Manipulating the cube, the proof of this condition is evident, and we consider it trivial.

2nd condition: Identity element. The existence of the identity element is verified by the movement of doing nothing on any of the 6 faces. With this, we ensure the identity I of any of the six faces of the cube or of any sequence of movements.

3rd condition: Inverse element. The inverse element in the Rubik's Group R means undoing the sequence of one or more movements performed. Here, there are two important considerations to be made: whether the movement was of only one face or a sequence of movements. Let X be a generic movement, a 90° clockwise rotation of one of the 6 faces; then its inverse element will be the movement, also of 90°, of the same face, but counterclockwise. This inverse element can be denoted as X^{-1}. If a sequence of movements is executed, then the inverse element will be the execution of the inverses of the movements but in the reverse order.

A sequence of movements is described by the sequence of codes for the rotations one wishes to perform, written from left to right in the same order in which the operations should be carried out, taking as reference the face in front of the manipulator (Chen, 2004; Joyner, 2008).

The inverse element in a group is unique, which is why, in the case of the Rubik's Cube, it is necessary for it to be well-defined to avoid duplicity when seeking the inverse element of a movement or a combination of movements. For this reason, we must be clear that performing only one movement is a particular case of performing a sequence of movements, ensuring the uniqueness of the identity element within the Rubik's Group R.

Despite the commutative property holding true between movements on opposite faces, the same does not apply to movements on adjacent faces. Movements FR and RF are not equal, as shown in Figure 3. This leads us to conclude that the Rubik's Group R is not abelian, making the cube, in particular, a challenging puzzle to solve.

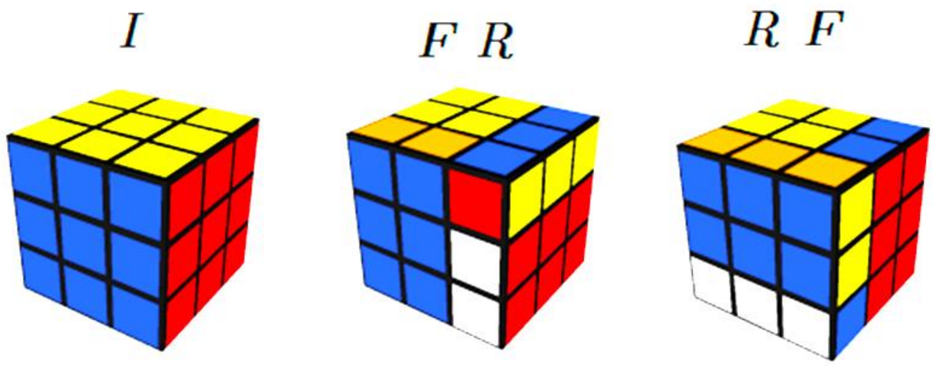


Figure 3. FR and RF sequences and their differences

Table 2
Square symmetries, adapted from Romero (2013)

	I	R	R^2	R^3	V	H	D_1	D_2
I	I	R	R^2	R^3	V	H	D_1	D_2
R	R	R^2	R^3	I	D_1	D_2	H	V
R^2	R^2	R^3	I	R	H	V	D_2	D_1
R^3	R^3	I	R	R^2	D_2	D_1	V	H
V	V	D_2	H	D_1	I	R^2	R^3	R
H	H	D_1	V	D_2	R^2	I	R	R^3
D_1	D_1	V	D_2	H	R	R^3	I	R^2
D_2	D_2	H	D_1	V	R^3	R	R^2	I

Symmetries in the square

To understand the permutations that can occur on the cube, we analyze the basic concepts of permutation groups with the support of a simpler element: the symmetries of the square, based on Romero (2013) and Davva (2021).

Let's consider the movements of the square as a rigid body. Figure 4(a) represents a 90° clockwise rotation, denoted by R (for rotation). Figure 4(b) corresponds to the symmetry about a vertical axis passing through the center of the square, denoted by V (for vertical symmetry):

There are two ways to understand the permutation of the letters A, B, C, and D on the square. We can think of the action as "moving to" or "being replaced by." We can also consider that the permutation acts on the initials or symbols, regardless of their position, or that it acts on the content of the positions, regardless of the symbol currently occupying that position. These distinctions are important when multiplying permutations. We can represent the previous permutations with these two criteria, as seen in Figures 4a and 4b. Thus, in Figure 5, we have the representation of two permutations with these two criteria. We have all the symmetries of the square. Consider: I – identity, R – rotation, V – vertical, H – horizontal, and D – diagonal (Romero, 2013) as presented in Table 2.

In the following section, we provide a brief overview of the possibilities for exploring the symmetries of the square and the Rubik's Group in GeoGebra.

Permutation groups, Rubik's group, and GeoGebra

In the study of Group Theory, particularly permutation groups, we can observe that there are relationships or combinations that can be visualized concretely, which facilitates the understanding of the subject (Zazkis et al., 1996; Monteiro et al., 2019). An example can be illustrated in Figures 6 and 7, depicting models of square symmetry groups, the basic figure of our cube, projected in the GeoGebra software:

The possibility of visualizing and manipulating this type of group using software like GeoGebra allows for understanding its relationships and properties, identifying symmetries, and grasping possible applications of the subject (Alves, 2019; 2022). A permutation group, also known as a symmetric group, is a group formed by all possible permutations of a finite set of elements. We

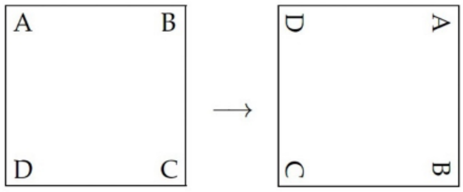


Figure 4(a)

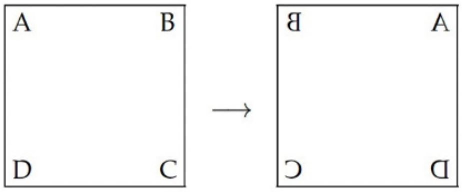


Figure 4(b)

Figures 4 (a and b). Symmetries of a square

$$
\begin{array}{c|cccc|cccc}
 & & & move & & & & replace & \\
 & A & B & C & D & A & B & C & D \\
R = & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & B & C & D & A & D & A & B & C \\
 & A & B & C & D & A & B & C & D \\
V = & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
 & B & A & D & C & B & A & D & C
\end{array}
$$

Figure 5. Representation of permutations with the criteria

explore aspects of its definition based on Wussing (1984), Joyner (1996), Davis (2006), and Garcia and Laquin (2002). In particular, we present Theorem 1, as stated in Joyner (1996):

Theorem 1. Let X be a finite set. Let g_1, g_2, ..., g_n be a finite set of elements of permutations of X (such that all belong to S_X). Let G be the set of all possible products in the form:

$$g = x_1 * x_2 * \ldots * x_m, m > 0 \qquad (1)$$

where each of x_1, ..., x_m is taken from the set $\{g_1, \ldots, g_n\}$. The set G, along with the group operation given by the composition of permutations, is called the permutation group generated by $\{g_1, \ldots, g_n\}$. Sometimes we write $G \ = < g_1, \ldots, g_n > \subset S_X$ (Joyner, 1996, p. 90).

Proof: Lemma 1. A permutation group is a group

Let G be a permutation group, as indicated in Theorem 1. We can prove that each $g \in G$ has an inverse element. The set $\{g^n / n \geq -1\} \subset S_X$ is infinite. Thus, there exist $n_1 > 0, n_2 > n_1$ such that $g^{n_1} = g^{n_2}$. Therefore, $g^{-1} = g^{n_2 - n_1 - 1} = 1$. ■.

The proof is trivial, and the remaining properties can be developed by the reader.

The permutation group of a finite set X, denoted by S_X, consists of all possible permutations of X. The order of this group is $n!$, where n is the number of elements in X, and $n!$ represents the factorial of n, which is the product of all integers from 1 to n (Garcia & Lequain, 2002). With the notion of Permutation Groups, it is possible to establish isomorphisms between Symmetric Groups and Rubik's Subgroups. This combination of these two concepts allows the analysis and creation of sequences of movements to solve the cube.

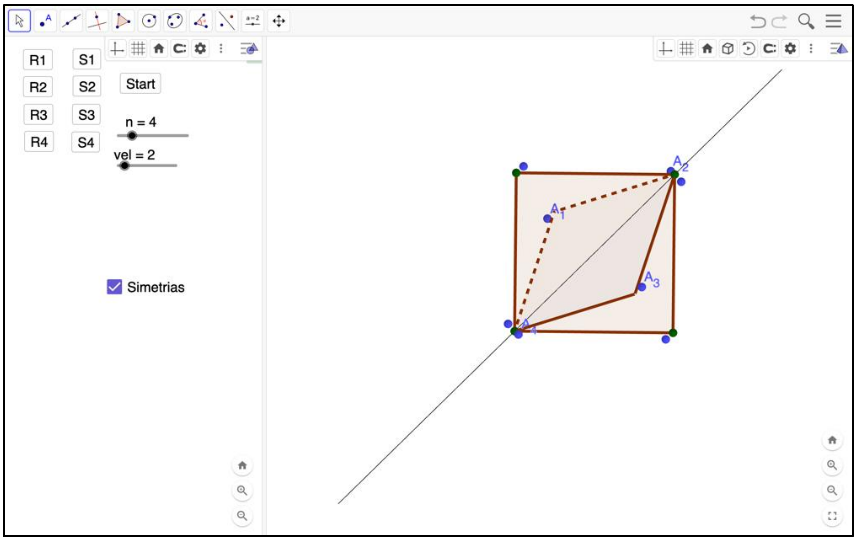


Figure 6. Symmetry of the square in the GeoGebra software (diagonal rotation)

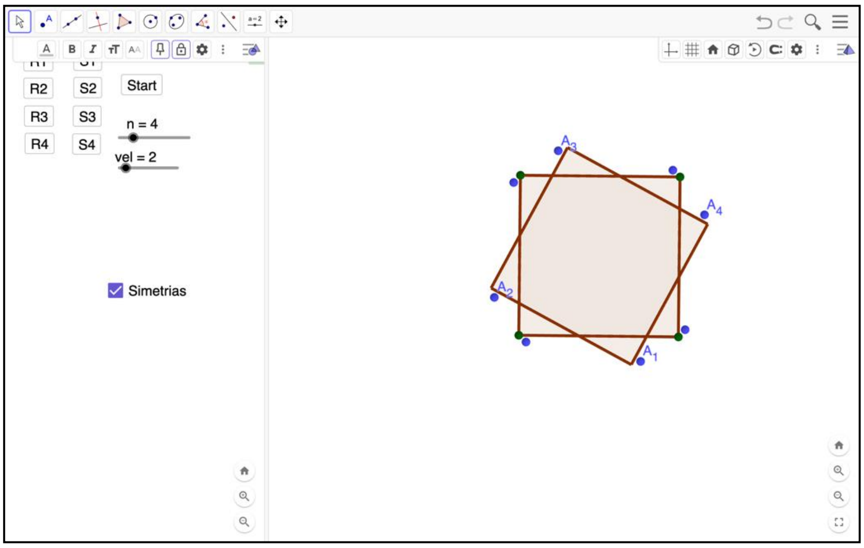


Figure 7. Symmetry of the square in the GeoGebra software (rotation at the vertex)

Expanding on the concepts related to the symmetry of the Rubik's Cube, we have that the set of all permutations of Z_n is denoted by S_n and is called the symmetric group with n letters (Schultzer, 2005). When solving the Rubik's Cube, users often perform various movements of the following type: make a move, say M_1, then another move M_2, and then make the inverse of the first move, ${M_1}^{-1}$. For example, the sequence $(R^{-1}D^2RB^{-1}U^2B)^2$ consists of such movements. This movement is a twist of two corners: the *URF* corner is rotated once clockwise, and the *BLD* corner is rotated once counterclockwise.

To understand how to calculate the number of movements or sequences of movements of the cube until an identity is obtained, it is necessary to comprehend the concepts of the order of a group, cycles, and cycle product. Given the brevity of the manuscript, we will use Proposition 1 in an abbreviated manner (Garcia & Laquin, 2002):

Proposition 1. Let μ_1, ..., $\mu_n \in S_n$ be disjoint cycles of lengths l_1, ... l_n, respectively. The order of the product $\mu_n \ldots \mu_1$ has an order equal to the least common multiple of (l_1,... l_n).

Example: Let μ = (1 2)(3 4 5) be the disjoint product of a 2-cycle and a 3-cycle. Since the order of (1 2) is 2 and the order of (3 4 5) is 3, it follows that $O(\mu) = lcm(2,3) = O(\mu) = 6$.

In the case of the Rubik's Cube, consider the following example: Let the sequence $S = F^4$. By repeating the F movement 4 times, we return to the identity, as shown in Figure 8.

In this case, the order of S is 4, i.e., $O(S) = 4$. All movements (UU^{-1} $DD^{-1}RR^{-1}LL^{-1}FF^{-1}BB^{-1}$) of the faces, both clockwise and counterclockwise, are of order 4.

It is possible to calculate the order of a sequence of movements, but to arrive at this value, we need to observe on the Rubik's Cube itself which small cubes will be affected by the change in position caused by this sequence of movements. By executing the movement sequence one or more times, we notice that all the smaller cubes that move occupy only certain positions, meaning that during the permutation, they are not all interchanged with each other. Thus, it can be observed that these permutations occur in groups that form a cycle, where only the small cubes belonging to the same cycle are permuted among themselves.

Indeed, we notice that the cubes UB, UR, and FR permute among themselves, similarly, the cubes LUB and RUB permute among themselves, and finally, the cubes RUF and RDF permute among themselves. Therefore, we can write, in cycle notation, the sequence S that causes the following permutations:

$$S = (UB\ UR\ FR)(LUB\ RUB)(RUF\ RDF) \tag{2}$$

composed of one 3-cycle and two 2-cycles. This example is illustrated in Figure 9. You can also associate these numbers as follows:

$$\begin{array}{ccccccc} UB & UR & FR & LUB & RUB & RUF & RDF \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 4 & 6 & 1 & 3 & 5 & 7 \end{array} \tag{3}$$

and, in this sense, we can write the permutation of the sequence S as:

$$S = (2\ 4\ 6)(1\ 3)(5\ 7) \tag{4}$$

The lengths of the cycles $(UB\ UR\ FR)$, $(LUB\ RUB)$, and $(RUF\ RDF)$ are, respectively, 3, 2, 2. Therefore, the order of S is given by the least common multiple (lcm) of 3, 2, 2. Thus, $O(S) = lcm(3, 2, 2) = 6$. We can also infer that the permutation generated by S is even, as the sequence S can be written as a product of an even number of transpositions, for example: $S = (2\ 4)(2\ 6)(1\ 3)(5\ 7)$.

The possibility of exploring the Rubik's Cube in GeoGebra can provide a better understanding of permutation groups and other related topics. In a non-exhaustive manner, we can illustrate with the construction shown in Figure 10, based on a publication in the Spiegel magazine (1981) and on the Rubik's Cube homepage (https://rubiks.com/en-US/) itself.

In the foreground, the Rubik's Cube is made up of 27 smaller individual cubes, which together form a large 3 x 3 x 3 cube. However, in the real situation, we only have 21 moving parts, namely 1 axis system (with 6 single-color fixed center pieces), 8 three-color corner pieces and 12 two-color border pieces. Spatial movement with the software allows for better understanding (Figures 11 and 12). The buttons with arrows are intuitive and indicate the color to be moved and the direction of the movement. The 'back/colors' button performs the inverse movements, and the 'move' buttons move the corners and centers as many times as they are pressed, following the directions indicated by the arrows.

For mathematical understanding through subgroups of the Rubik's Group, formed by sequences of steps and movements, and the development of a strategy that uses the least possible memorization, i.e., types of movements to be performed, concepts of commutators and conjugates are employed. This method involves few combinations of sequences, which, in turn, are intuitive. The use of these concepts is commonly found in more sophisticated solving methods, where the goal is to optimize the number of movements required to solve the cube (Bandelow, 1982; Travis, 2007; Joyner, 2008).

A commutator is a sequence of movements that has the net effect of performing a single specific operation on the cube, consisting of three parts: (i) an operation (X), a second operation (Y), and the inverse of the first (X'). The notation for a commutator is $[X, Y] = XYX'Y'$. In a simplified way, a commutator is a way to swap two pieces or sets of pieces without affecting the rest of the cube.

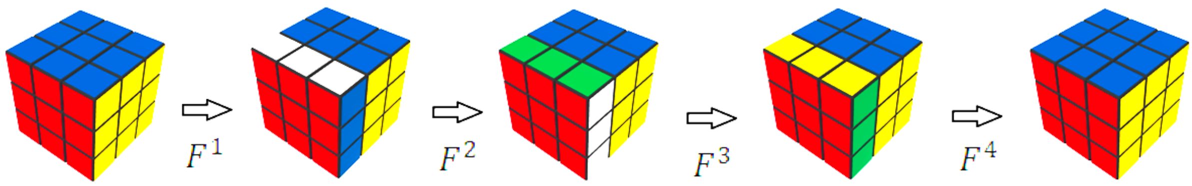


Figure 8. Sequence $S = F^4$.

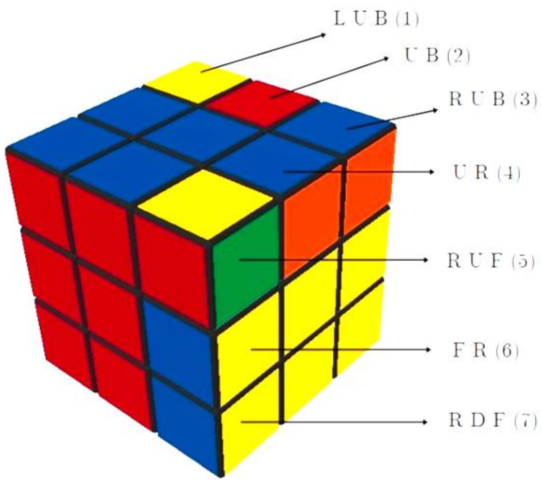


Figure 9. Cubes affected by the S sequence

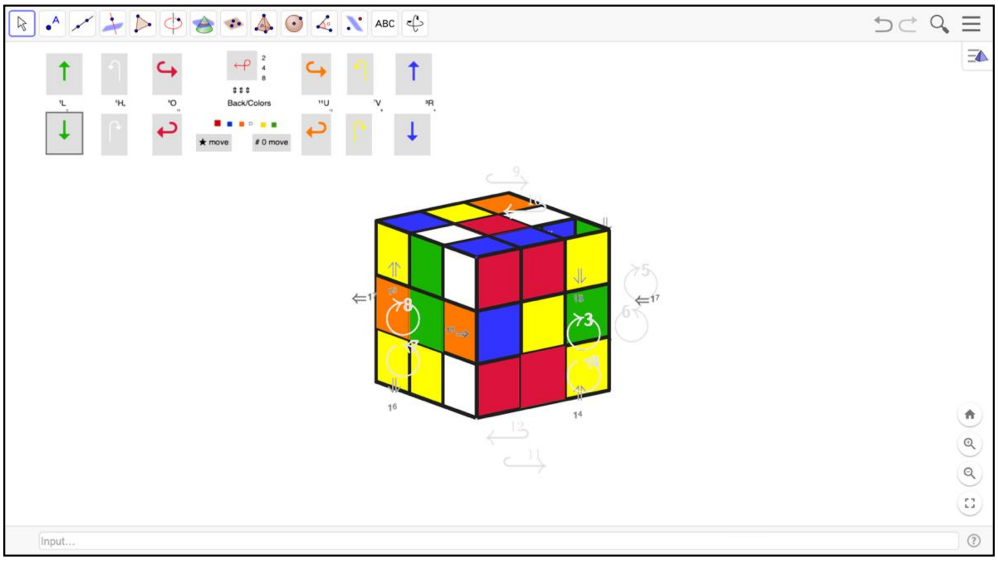


Figure 10. Construction of the Rubik's cube in GeoGebra.

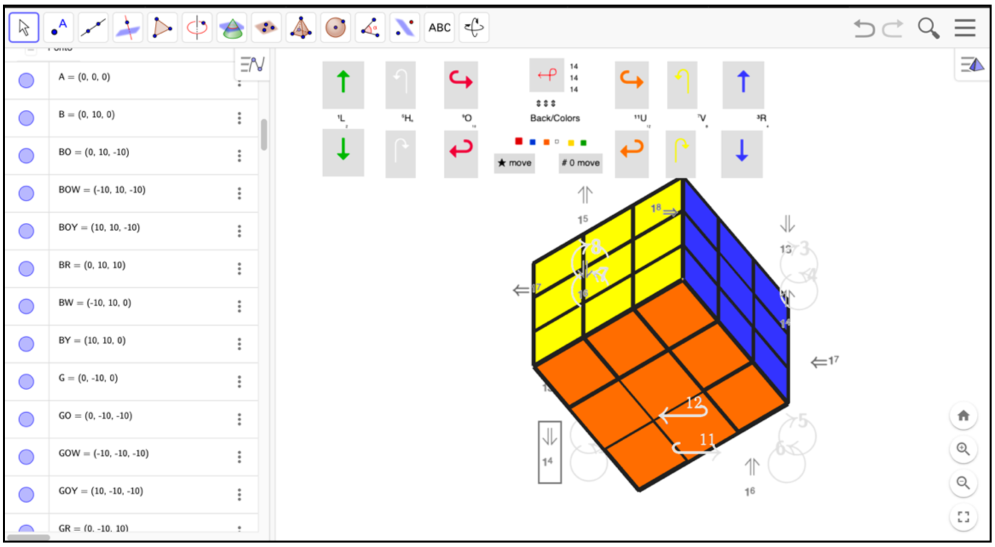


Figure 11. Rubik's Cube in GeoGebra

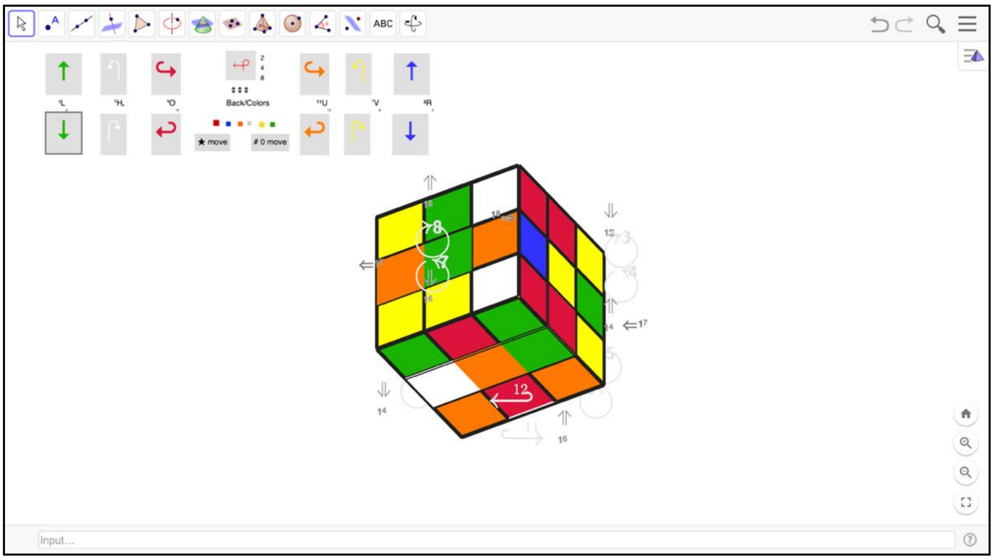


Figure 12. Shuffle of the smaller faces of the Rubik's Cube

On the other hand, the conjugate is related to using an algorithm X to change the state of a piece, applying X, followed by an operation Y, and finally the inverse of X (X'). The result is a new algorithm that performs the same operation in a different position. Formally, if X is an algorithm (sequence of movements) and Y is another algorithm, then the conjugate of X by Y is given by YXY'. Thus, for solving the Rubik's Cube, conjugates are used to apply algorithms in specific situations, altering the orientation or position of pieces without affecting other parts of the cube. This model, at first, can be explored in the software or even with the physical object, seeking to understand the mathematical concepts associated with the movements, as presented in Table 3.

However, in each stage, it will generally be necessary to repeat each sequence several times until the desired result is achieved, making the method slow and involving many movements. Despite this, its repetition allows for a better understanding of the combinations of movements and logic associated with the executed sequences. The expected final result can be illustrated in the software, as shown in Figure 13.

The results show that using GeoGebra to model and manipulate the Rubik's Cube facilitates the understanding of permutation group concepts, especially through the visualization of cycles and

Table 3
Solution strategy with steps, objectives, and necessary sequences.

Step	Objective	Procedure
1	Form a cross on one of the faces.	Performed intuitively.
2	Arrange the 4 edge cubes, in the median layer, of the faces adjacent to the one chosen in the 1st step.	$U^{-1}F^{-1}UF$ *or* $UFU^{-1}F^{-1}$
3	Arrange the cubes on the opposite side of the face chosen in the 1st step, forming a cross on all six faces of the Rubik's cube.	$(RU^{-1}R^{-1}U)(F^{-1}UFU^{-1})$ *or* $(U^{-1}F^{-1})(U^{-1}F)(U^{-1}F^{-1})(U^{-1}F)U^{-1}$
4	Position all 8 corner cubes correctly	$R^{-1}(ULU^{-1})R(UL^{-1}U^{-1})$ *or* $(ULU^{-1})R^{-1}(UL^{-1}U^{-1})R$
5	Orient all 8 corner cubes correctly.	$(LD^2LF^{-1}D^2F)U(F^{-1}D^2FLD^2L^{-1})U^{-1}$ *or* $(F^{-1}D^2FLD^2L^{-1})U^{-1}(LD^2L^{-1}F^{-1}D^2F)U$

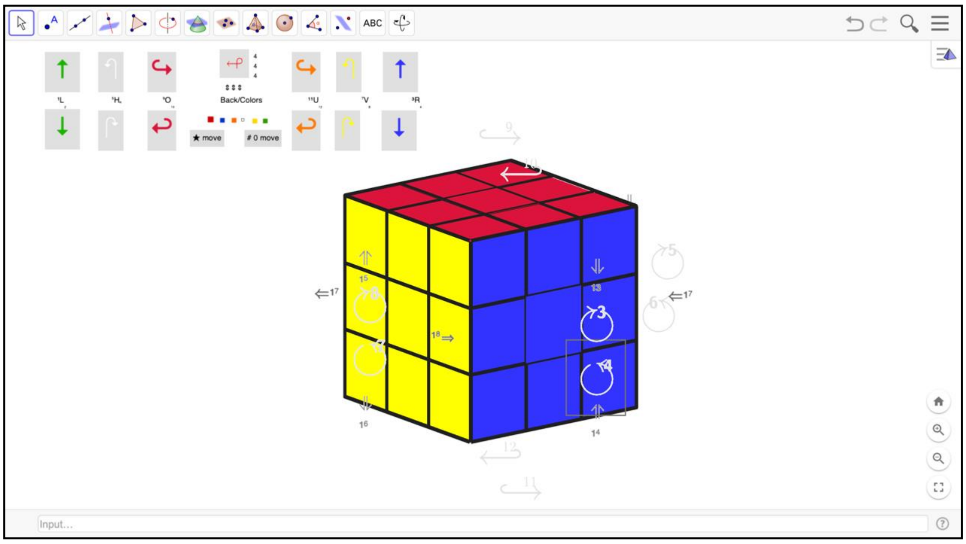


Figure 13. Rubik's Cube solved in GeoGebra

symmetries. For example, when applying the algorithm $S = (RUR'U')$, which is common in solving the Rubik's Cube, GeoGebra was used to visualize the resulting permutations of the smaller cubes that make up the cube's face. It was observed that this specific sequence of movements resulted in a cyclic exchange of three corner pieces, demonstrating a cycle of order 3 within the permutation group.

Furthermore, when exploring the sequence $S = (FRUR'U'F')$, the software demonstrated how this movement produces a permutation cycle involving two edge pieces, illustrating a cycle of order 2. The visualization of these operations in GeoGebra allowed students to clearly identify how the permutations affect the cube's configuration, something that can be difficult to understand without visual support.

These results reinforce the idea proposed by Romero (2013), which highlights the importance of visualization for understanding permutation groups in educational contexts. However, the study presents some limitations, such as the absence of a quantitative evaluation of the impact of this method on student learning, suggesting the need for future studies to assess the effectiveness of using GeoGebra in the classroom.

Another aspect observed was GeoGebra's ability to demonstrate the non-commutative property of the Rubik's Cube permutation group. By applying the sequence FR (front face turn followed by right face turn) and comparing it with the sequence RF (right face turn followed by front face turn), students could visualize that the resulting configurations are different, clearly illustrating the non-abelian nature of the cube's movement group.

Therefore, the use of visual tools like GeoGebra not only facilitates the visualization of Group Theory concepts but also promotes a deeper understanding of the algebraic properties underlying the Rubik's Cube. These findings suggest that using GeoGebra can be a valuable approach to teaching abstract mathematical concepts, making them more accessible and intuitive for students.

There are many other ways to bring the cube back to its original position, and these are proposed by various authors, which can be explored using the software, considering its mathematical characteristics. Such strategies can be discussed when exploring more complex concepts, such as permutation groups and the particular case of the Rubik's Group. However, given the brevity of our manuscript, we provide access to the construction as a study proposal for enthusiasts of this toy, available at: https://www.geogebra.org/m/ft4dwxvb.

CONCLUSIONS

The mathematics of the Rubik's Cube involves understanding combinatorial ideas such as permutations and counting arguments. Group Theory is crucial for analyzing and comprehending the mathematical properties of the Rubik's Cube, as each combination of movements forms a group, and the study of these groups enables an understanding of their structural properties, such as the ability to solve any configuration using a specific sequence of movements.

Permutation groups are applied in the development of efficient solving methods for the Rubik's Cube, such as the Layer-by-Layer Method (or CFOP method - Cross, F2L, OLL, PLL), which utilizes the mathematical principles of permutation groups to simplify the solving process of the Cube. Thus, the relationship between the Rubik's Cube and permutation groups lies in the mathematical representation of the cube's configurations as permutations and the analysis of the groups generated by possible movements.

Visualization plays a significant role in the mathematical understanding of the Rubik's Cube. Some of its contributions include enhancing intuition and spatial understanding, recognizing patterns and symmetries, applying abstract theoretical concepts in a practical manner, and optimizing problem-solving abilities. In the case of the Rubik's Cube, technology can enhance this understanding through virtual simulation, modeling, and graphical representation of interactive features.

The use of GeoGebra can provide visual and algebraic support, offering a differentiated approach to understanding the topic. It is possible to explore graphical representation, movement animations, permutation matrices, and the study of group theory in abstract algebra with the software. The visualization possibilities with the software and the exploration of its interface to comprehend the construction and permutations of cube movements can be an approach for mathematics education instructors, utilizing a tangible object and relating this practical example to real-world applications.

This study highlights the potential of GeoGebra as an effective tool for visualizing and understanding permutation groups, contributing significantly to the teaching of complex concepts in Group Theory. We recommend that educators explore the use of visual tools in their teaching practices to facilitate students' understanding and we suggest that future research investigate the impact of these technologies on mathematics learning.

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AUTHOR'S DECLARATION

Authors' contributions	RTS: conceptualization; investigation; methodology; writing – original draft preparation; visualization; validation; writing – review

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