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# A framework for designing problem solving task for secondary school mathematics classroom

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**ABSTRACT**

Although problem solving has been playing a critical role in the teaching and learning of mathematics in K-12 in many countries in the world, anecdotal evidence from mathematics classrooms shows that teachers are still facing challenges in designing mathematical tasks on problem solving for classroom instruction. Despite the effort of several existing research studies on enacting problem solving in the mathematics classroom, the guiding principles for designing problem solving tasks remain largely unexplored, thus teachers are left without being empowered to design their own problem-solving tasks. The objective of this paper is to provide a comprehensive framework on designing problem solving tasks through a list of interrelated guiding principles. The proposed framework, which we name MIRACLE, foreground seven key considerations which mathematics educators should take into consideration when designing a task for a problem-solving lesson: Mathematical content, method of Instruction, Required knowledge, Assessment, Complexity of problem, Learner's profile, and Enactment of lesson. This paper also provides two exemplars on how the framework could be used to design mathematical tasks for problem solving *through* problem solving

## INTRODUCTION

Problem solving has been the heart of the mathematics curriculum in many countries around the world, after the seminal work by Polya (1945). Researchers have noted that developing students' ability in problem solving has since then been part of the learning objectives in school mathematics curricula since the 1970s in many countries around the world (Fan & Zhu, 2007).

Despite the emergence of mathematical problem solving in school mathematics curricula since many decades ago, research literature up to the last decade still shows that teachers generally are still struggling with the dichotomy between developing fluent basic skills and problem solving ability in the classroom (Kaur & Yeap, 2009). Researchers have not surprisingly started to explore approaches to enact mathematical problem solving in the authentic mathematics classrooms (Toh et al., 2008a; 2008b; 2013). The approaches to enacting problem solving in the classrooms, as introduced by the researchers are based on sound conceptual frameworks of Polya (1945) and Schoenfeld (1985). Researchers have also recognized the difficulties such enactment brings. Hence, to transform problem solving into an authentic part of the enactment in the mathematics classroom faces several two main challenges among many others:

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Firstly, mathematics teachers are often required to only modify challenging tasks in order to engage their students cognitively and, to a lesser extent, affectively. They were usually not required to create an original problem to be used for problem solving. Thus, they might lack insight to the mathematical and theoretical considerations that underlie the design of the problem used for problem solving. This lack of full participation in task design may result in limiting the affordances of the problem.

Secondly, teachers are usually expected to cover the syllabus content in a timely manner. In many countries, teachers are required to help students succeed in high-stake national examinations. Hence, allocating additional curriculum time to problem solving is unrealistic. The additional time that may be required to teach problem solving on top of the original tight schedule further contributes to the resistance towards the teaching of problem solving in classrooms. Thus, some researchers (Leong et al., 2016) proposed the *infusion* of problem solving into the lesson as a more viable approach of teaching problem solving.

As an attempt to address the two main challenges described above, in this paper we propose a framework that can serve as a guide for educators and teachers in designing problem solving tasks that can be infused into the authentical classroom instruction. Through this proposed framework, we aim to empower the classroom teachers to be designers of problem-solving tasks – an approach quite different from the above discussion in which teachers *adapt* tasks from existing resource created by researchers. Furthermore, the proposed framework focuses on another paradigm of problem solving, that of teaching *through* problem solving, instead of teaching *about* problem solving, which requires much additional curriculum time, or teaching *for* problem solving, which is the usual mode of practice in existing mathematics classrooms.

## METHODS

In this study, we conducted an extensive literature review of education literature on the following areas: (1) classical learning theories (Piaget, 1952; Vygotsky, 1978) and the later developments based on these classical learning theories (Hino, 2018; Matlin, 2014; Tan et al., 2017); (2) mathematics tasks and a framework on mathematics task (Mason & Johnston-Wilder, 2006; Stein et al., 1996) and later studies on mathematical tasks (Choy, 2018; Toh, 2010); (3) seminal works on mathematical problem solving (Lester, 1983; Polya, 1945; Schoenfeld, 1985); (4) enactment of problem solving in authentic mathematics classrooms (Toh et al., 2008a; 2008b; 2011; 2013).

### Mathematical task framework

We note the distinction between a mathematical task and activity. (Mason & Johnston-Wilder 2006) defined mathematical tasks as instructions given to students to initiate mathematical activity, which is the engagement students have with a mathematical idea. It is important for teachers to consider the mathematical activities afforded by the tasks during the design and implementation of the tasks in class (Choy, 2018).

The mathematical tasks framework was first proposed by Stein et al. (1996) to investigate the relationship between teacher instruction and student learning. The framework (Figure 1) delineates the stages through which teachers' treatment of mathematical tasks can impact student learning. In setting up a mathematical task, decision has to be made on how the task as represented in instructional materials can be presented to the students. Factors such as the teacher's goals, content knowledge, and knowledge of students could impact the stage on setting up the task. The link between mathematical tasks and the corresponding mathematical activity were examined in terms of task features and cognitive demands.

Stein et al. (1996) defined task features as the aspects of tasks that mathematics educators consider critical for engaging students' thinking, reasoning, and sense-making (such as the existence of multiple-solution strategies, the extent to which the task lends itself to multiple representations, and the extent to which the task demands explanations and/or justifications from the students), while cognitive demand refers to the types of thinking processes used in completing the task (such as, memorization, the use of procedures and algorithm, and the use of complex thinking and reasoning strategies). The factors that could affect such implementation include classroom norms, task conditions, teacher instructional and student learning habits and dispositions.

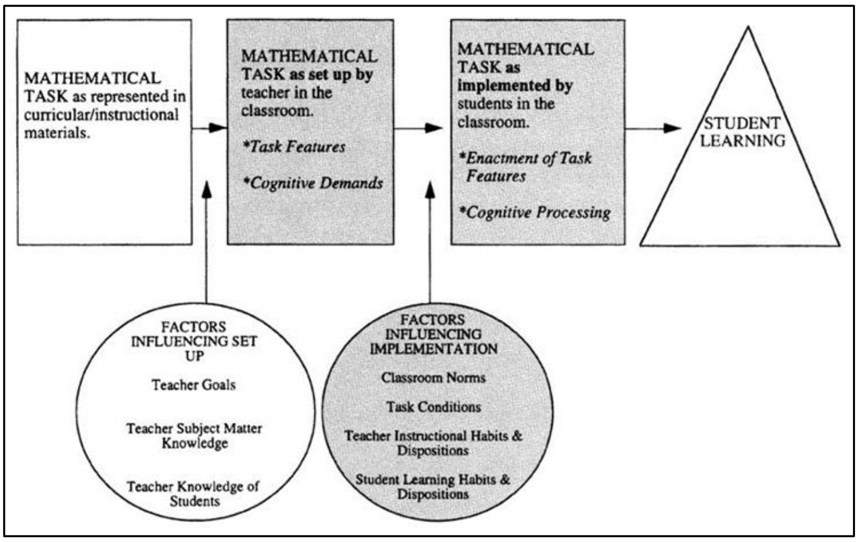


Figure 1. Mathematical tasks framework (Stein et al., 1996)

Stein et al. (1996) found that during the task implementation stage based on the proposed framework, the task features remained consistent with how the tasks were set up, but the cognitive demands of high-level tasks tended to decline. The factors associated with the change in cognitive demands between the set-up phase and the implementation phase include the trivialization of the challenges of the original task, inappropriateness of task for students, shift of focus from the processes to correct answers, time constraint, lack of accountability, classroom management problems, among other factors.

## Mathematical problem solving

Polya (1945) defined a problem as a task for which the solution path is not immediately apparent. Building on Polya's definition, Lester (1983) added an affective dimension to the definition. He stated that for a task to be considered as a *problem*, an individual or group must be motivated to find a solution and/or attempts to do so.

Other researchers also believe that problem solving should be the key business of the enterprise of mathematics education. For example, Schoenfeld (1983) believed that "*the primary responsibility of mathematics faculty is to teach their students to think: to question and probe, to get to the mathematical heart of the matter, to be able to employ ideas rather than to regurgitate them*" (p. 2). He recommended that problem solving lessons be integrated in standard mathematics curriculum in order to achieve that goal.

To engage students in mathematical problem solving, it is crucial for students to acquire a problem-solving model. This is especially important in the event that the process of solving the problem is not smooth. Any proficient mathematician would have built up their own model of problem solving. Polya (1945) proposed a four-step model for problem solving: understanding the problem, devising a plan, carrying out the plan, and looking back.

As noted by Schoenfeld (1985), while he applauded Polya's model of problem solving, he also highlighted that merely equipping students with a problem solving model such as Polya's does not necessarily turn them into good problem solvers who are capable of exhibiting problem solving behaviors similar to those of professional mathematicians. He proposed four aspects of mathematical thinking that can contribute to problem solving behaviour: resources, heuristics, control, and belief systems.

Schroeder and Lester (1989) introduced three paradigms of problem-solving instruction: teaching about, for, and through problem solving. These conceptions still remain useful as descriptions of enactments of mathematical problem solving in classrooms (Leong et al., 2016). Teaching *about* problem solving uses problems to teach students problem solving skills (Polya's four-

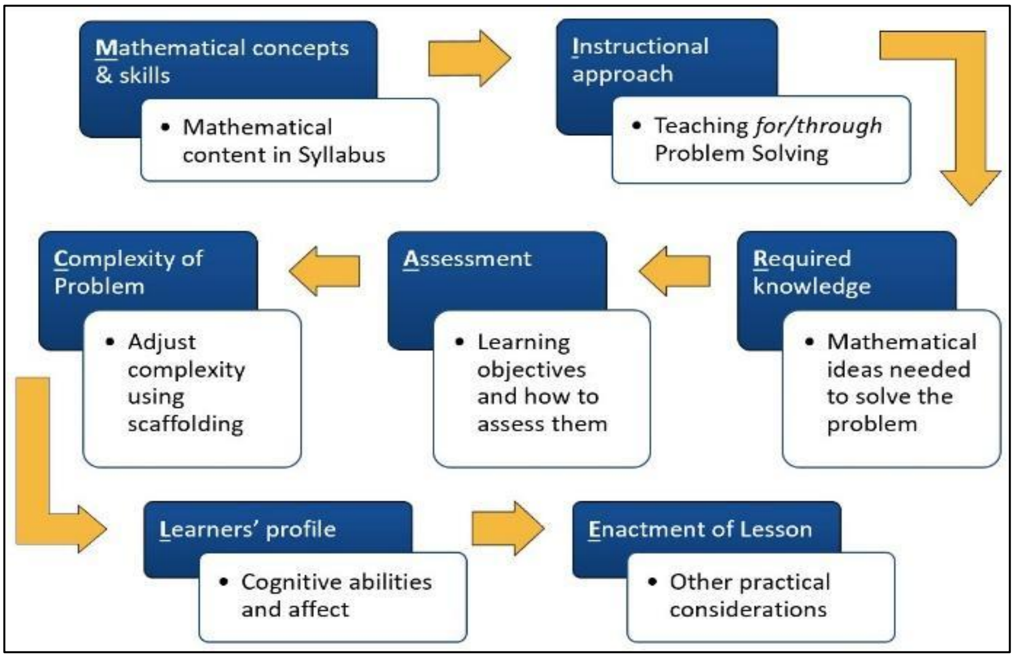


Figure 2: The MIRACLE Framework for Designing Problem Solving Tasks.

step model, heuristics, problem solving disposition) that they can use to solve problems. An example of an enactment demonstrating teaching about problem solving is the mathematics practical paradigm advocated by Toh et al. (2008a; 2008b; 2011; 2013).

The typical classroom instruction, if problem solving is considered as incorporated in the curriculum, uses the mode of teaching *for* problem solving. In this mode, the students are taught mathematical knowledge before they are posed problems for which the solution requires the mathematical knowledge taught (Van de Walle et al., 2019). In teaching *through* problem solving, the teachers engage their students in problem solving processes with the goal of learning mathematics content (Lester & Charles, 2003).

In short, the mode of problem solving is determined by the roles of the tasks used by the teacher. The teacher can use the task to teach students problem solving skills as the objective (teaching *about* problem solving); or used for students to acquire new concepts so that they are able to solve problems later (teaching *for* problem solving); or for students to go through problem solving processes in order to acquire new content knowledge (teaching *through* problem solving).

## FINDINGS AND DISCUSSION

Based on the literature review that we have conducted and together with our collective experience in classroom teaching, we propose the MIRACLE framework (Figure 2) for the design of problem-solving tasks. In the framework, we propose seven design considerations, which form the foundational structure of the framework. These seven considerations are the key features that classroom teachers should take into consideration when designing problem solving tasks to be used in a mathematics classroom. The important message that the MIRACLE framework highlights is that the designer of a problem-solving task does not end with the delivery of the task. The whole range of considerations, starting with the mathematical concepts and skills, and ending with a plausible enactment in the classroom, need to be deliberated.

### Mathematical concepts and skills

The mathematical concepts and skills afforded with a task is the first key elements that distinguishes teaching *for/through* problem solving from teaching *about* problem solving. Prior to designing the mathematical task, the designer of the task first determines the learning objectives of the task, which are usually based on the syllabus content (Stein et al., 2006). The nature of learning objectives is dependent on the paradigm of problem solving that one engages in. In teaching *about* problem solving, the content serves to forward the problem-solving processes, the acquisition of which form the learning objectives. Thus, in this paradigm, the mathematical concepts and skills only serves to foreground the processes, so the choice of the concepts is less important. On the other hand, in teaching *for* problem solving, the mathematical concepts and skills form the core learning objectives; The learning objectives of teaching *through* problem solving include both mathematical concepts and skills, and the processes.

For tasks designed for teaching for/through problem solving, the mathematical concepts and skills that are required to solve the designed problem task and its possible extensions should align with the mathematical concepts and skills in the learning objectives. In addition to the concepts and skills aligned to the learning objective (or learning outcomes), the designed task should also provide the students to use mathematical ideas beyond the scope of the learning objectives. Thus, it is crucial that the designed problem task be mathematically rich for the learning objectives and their extensions to be achieved.

## Instructional approach

In the next stage of deciding the instructional approach, the *instructional* objective needs to be determined after the learning objectives have been identified by the teacher. Wong (2015) distinguishes between the terms learning objective and instructional objective. "*Learning objectives (learning outcomes) are targets for student learning, while instructional objectives (or specific instructional objectives, abbreviated as SIOs) are the purposes of teacher teaching*." (Wong, 2015). Typically, instructional objectives must be observable and measurable (Butt, 2008). Thus, instructional objectives are characterized by the use of action verbs found in Bloom's revised taxonomy (Anderson & Krathwohl, 2001), which covers the cognitive, affective, and psychomotor domains (Houff, 2012). With the instructional objectives having been identified, teachers next decide the appropriate instructional approaches suitable to achieve the concepts and skills, teaching for/through problem solving as it would impact the setting of instructional objectives.

The instructional objectives vary according to the paradigm of problem solving when teaching *for* problem solving, the instructional objectives describe the students being able to apply the mathematical content and skills in the learning objectives in solving the problem; the instructional objectives when teaching *through* problem solving would be for students to acquire the mathematical content and skills using the processes of solving the problem. It is worthwhile to note that the same problem could be used for different paradigms of problem solving. Hence, it is crucial that the teacher makes the distinction between these instructional approaches and decide which they would like to use when designing the problem.

## Prior knowledge or schema

Piaget (1952) proposed that *schema*, or prior knowledge, is the basic unit necessary for mental organization and mental functioning. A schema is an organized and generalizable set of knowledge about a certain concept, that can be applied to any instance of that concept (Matlin, 2014). The process of modifying an existing schema to understand new information is known as accommodation. When a student faces a new encounter which disrupts the balance between them and their environment, the process of modifying the existing schemata is carried out to restore equilibrium. Taken from this perspective, the designed problem should create a state of disequilibrium in the students, thus motivating them to activate prior schemata and undergo accommodation (Tan et al., 2017).

It is also important to consider the prior knowledge that students would require to solve and extend a given problem. Teaching for problem solving requires students to understand the mathematical ideas needed to solve the problem before attempting it. On the other hand, problems designed for teaching through problem solving should allow students the opportunity to learn new mathematical concepts or skills through the processes of solving the problem and/or its extensions.

Vygotsky (1978) defines the zone of proximal development as the area between what a student can do on their own and what they can do with adult guidance or in collaboration with more capable peers. Under the paradigm of teaching through problem solving, the designed problem should make use of students' prior knowledge and the processes of problem solving should allow students to proceed through their zone of proximal development and lead them to the new mathematical ideas.

Teachers play a key role in classroom interaction and discourse by "*scaffolding students' ideas so as allow them to extend and move forward as well as initiating, focusing and highlighting new mathematical ideas and thinking, and important mathematical practices*" (Hino, 2018). Teachers are expected to attempt to solve the designed problems using multiple methods, and consider all plausible extensions of the problem, in order to uncover the mathematical richness of the problem (Schoenfeld, 1983). This level of preparedness of the teachers could equip them sufficiently to investigate the relevant mathematical concepts and skills with their students and guide them towards the intended learning objectives if necessary. In order for teachers to be able to assess the richness of the problem, especially with the plausible extensions of the problem, teachers must have a firm grasp of subject matter knowledge, as asserted by researchers such as Hill et al. (2005).

## Assessment

Classroom assessment is used to gather information and provide feedback to support student learning and improve teaching practice (De Lange, 2007). Instruments for classroom assessment may be designed by the teacher or selected from other sources but must be locally controlled by the teacher and not an external party (Brookhart & McMillan, 2020). A common method to conduct classroom assessment is through the use of rubrics. Rubrics can be used to inform students about the learning objectives and provide teachers with clear guidelines to make the grading process less subjective (Depka, 2007). In the problem-solving approach developed by Toh et al. (2011), an assessment rubric was introduced accompanying the scaffolding of students' acquisition of problem solving (which they termed "*the practical worksheet*") was introduced. The scaffolding practical worksheet served to encourage students to use Polya's four-step model and can be used when teaching for/through problem solving. Note that the rubrics was designed to credit students for using mathematical processes involved in problem solving. The rationale of the accompanying assessment rubric was that if the processes (that is the processes of problem solving) are at least as important as the product (i.e., the solution of the problem), they must be assessed. This would reinforce the desirable problem-solving processes, which could then be internalized into habits. With this rationale in mind, teachers can design their own assessment rubric focusing on both processes and produce. For example, they can modify the assessment rubric in Toh et al. (2011) to include aspects of problem solving which they believe are valuable for their students.

## Complexity of the problem task

Yeo (2015) developed a framework that characterizes the openness of mathematics tasks based on five task variables: goal, method, complexity, answer, and extension. Problem solving tasks exhibit varying degrees of openness in terms of their solutions and extensions. This openness allows for the last stage in Polya's four-step model (i.e., looking back phase). Although problem solving tasks are typically problems with a fixed answer, these problem-solving tasks can differ in their openness based on their task complexity. Yeo (2015) noted the following with regards to task complexity: To summaries, a task is closed if it is simple enough for the students. Otherwise, it is open if it is too complex for the students and there is not enough scaffolding in the task statement. There are two types of openness in terms of complexity: the first type is subject-dependent because the teacher can provide enough scaffolding to close the task; the second type is task-inherent because it is inherently not possible to provide enough scaffolding to close the task. (p. 185)

When teaching through problem solving, based on Yeo's classification of openness of mathematics tasks, teachers need to be cognizant of the type of complexity of the problem used. Problem solving tasks should not be "*closed*" in its complexity. Caution should be taken by teachers when providing scaffolding through the design of the problem or during the problem-solving process, especially if the complexity of the designed problem is subject-dependent. Based on cognitive load theory, the expertise reversal effect is the need to modify instructional methods and levels of instructional guidance to changing levels of learner expertise during a learning session (Sweller et al., 2011). In order to provide appropriate guidance, teachers need to consider how they can scaffold the problem-solving process for students when they are in the stage of designing the problem.


Figure 3: Exemplar 1 on finding one angle of a regular octagon (coin)

Task: Find angle $x°$ without using any measuring tool.

## Learners' profile

Learners' profile is the next important consideration in designing a problem-solving task for classroom instruction. Learners' profile can be examined via both cognitive and affective domains. Mathematical cognition is the process by which individuals come to understand mathematical ideas, and that individuals' cognitive abilities vary according to individuals (Gilmore et al., 2018). Learner's affect, as defined by Chamberlin (2019), as their confidence, attitudes, and emotions towards a subject. In the context of mathematical problem solving, being aware of and in control of one's emotions can enhance the likelihood of them successfully solving the problem (Chamberlin & Parks, 2020). To add to affect domain, student's attitudes and confidence towards problem solving is crucial to solving the designed problem.

Differentiated instruction can be used by teacher to respond to the cognitive and affective variances that are inevitable among students (Tomlinson, 2014). "*In a differentiated classroom, the teacher proactively plans and carries out varied approaches to content, process, and product in anticipation of and response to student difference in readiness, interest, and learning needs.*" (Tomlinson, 2001). Differentiation based on the learners' profile can occur in a problem solving lesson in two ways: (1) When teaching for/through problem solving, teachers can differentiate the process by altering the quantity of scaffolding based on student readiness and learning needs or modifying the problem to cater to student interest; (2) in the phase of looking back phase, students can be guided to extend / generalize the problem to the extent that interest them and are within their ability. A typical example of the richness of extending or generalizing a given problem is discussed in details in Toh et al. (2008a; 2008b) with a detailed discussion of problem solving tasks.

## Enactment of the lesson

In the phase of designing the task, teachers should have the enactment of the lesson in mind. Lesson enactment is dependent on "*students' interests and experience, instructional strategies, curriculum resources, teacher's pedagogical beliefs, practice and expertise, parental expectations, school organization, community and culture, high-stakes examinations, curriculum policies, and so forth*" (Deng et al., 2013).

Teachers could take reference from the possible instructional objectives associated with the problems, and the micro-instructional objectives within the lesson enactment. As an illustration, Kaur et al. (2021) identified cycles of instructional practice observed in the Singapore mathematics classroom which occurred in the D-S-R cycle:

1) Development [D]: Teacher develops concepts/ demonstrates skills/engages students in activities to explore concepts.
2) Student Work [S]: Teacher sets students work to do where students apply the concepts/practice skills.

3) Review of Student Work [R]: Teacher reviews student work, drawing the attention of the whole class to errors, misconceptions, correct solutions, good presentations, etc.

In the lessons that Kaur et al. (2021) observed, a typical lesson often comprises of one or more cycles of instruction, which comprises combinations of D, S and R, depending on the number of objectives. This D-S-R cycle is a potential lesson structure that the teacher designer of the problem-solving lesson can be mindful in designing the task.

## Two examples of application of the MIRACLE framework

In this section, we provide two exemplars of designing problem solving tasks by taking the MIRACLE framework. We illustrate the task set for the paradigm of teaching through problem solving. The content of the exemplars is based on a typical secondary mathematics syllabus.

## Exemplar 1. Finding one angle of a regular octagon (coin)

### *Mathematical concepts and skills*

The mathematical concepts related to this question are (Figure 3): (1) the angle sum of a regular octagon; (2) all the angles of a regular octagon are equal. Extensions of the above two key concepts elicited from this question could be: (3) generalized result for the angle sum of a regular *n*-sided polygon; (4) sum of the exterior angles of a polygon; (5) the sum of the interior and exterior angles of a non-convex polygon. These extensions are perceived as natural progressions of engaging students in problem solving in the phase of extending the problem.

### *Instructional approach*

Under the paradigm of teaching through problem solving, we assume that the students have not been formally introduced to the mathematical concepts listed above, teachers can choose to teach these concepts through problem solving. The solution of the problem and/or its extension lends itself to students fulfilling the learning objectives, which consist of the content and process goals.

### *Required knowledge.*

After an initial draft of problem, the teacher considers the mathematical prior knowledge required to solve this problem in acquiring new knowledge. In this task, the prior knowledge required of the students is about the angle sum of triangles, a prior knowledge that the students have acquired at the primary education level. This prior required knowledge does not overlap with the learning objectives, as the new knowledge (angle sum of octagon) is built upon the knowledge of angle sum of triangles.

### *Assessment*

After finalizing the problem task for the problem-solving lesson, the teacher delineates the specific instructional objectives they would like students to achieve and devise a method to assess these objectives. The following are instructional objectives which students are to achieve through attempting this problem:

1) Apply properties of triangles to solve the problem
2) Decompose a polygon into triangles to solve the problem
3) State that angles and side lengths of regular polygons are the same
4) State the angle sum of interior and exterior angles of any convex polygon

The first two instructional objectives in the above list are based on mathematical processes required to solve the problem, while the last two instructional objectives are the content objectives stated earlier. To assess student learning, teachers could use a rubric with these instructional objectives as criteria and rate student competency in each criterion on a scale.

### *Complexity of problem*

The following are some examples of scaffolding that can be provided to students through verbal guidance from the teacher:

1) Consider a simpler problem (i.e., regular polygon with less sides)
2) Have you seen something similar before (i.e., finding angle of triangles)
3) Can you make use of prior knowledge (i.e., angle sum of triangles)

The complexity of a problem can be adjusted by the amount of scaffolding provided by the teacher. Scaffolding should only be provided when necessary and should not "*close off*" (Yeo, 2015) the problem. The suggested guiding questions above prompts student to think in the right direction towards solving the problem without completely removing the challenge of the problem.

### *Learner's profile*

In designing this task, as in any other task, the teacher should be cognizant of the cognitive and affective needs of the entire spectrum of students. In this exemplar, the teacher can group their students based on their readiness levels and provide different groups with different levels of scaffolding depending on their cognitive needs, including providing varying amount of prior knowledge. The problem task provided in this exemplar, for example, could be either presented using a pentagon or a regular octagon. The form presented in this exemplar makes use of real-world context, with the objective to lower students' anxiety in approaching the problem, and to make the content appear relevant to their daily life.

### *Enactment of lesson*

In designing a task to be used for teaching through problem solving, the teacher needs to be cognizant of the lesson enactment. The teacher could consider the D-S-R cycles in typical lesson enactment to provide a guide how the lesson could be enacted. As an illustration, the types of activities related to each sub-phase.

1) Development (D): Teacher provides an introduction of the concepts (in this example, the definition of a polygon, recall of angle properties of triangles) and raises to the students' cognizance of mathematical problem-solving processes. Then teacher sets the task for the students (without providing a complete solution for the students).
2) Student work (S): Students are engaged to solve the problem using their problem solving processes that they have acquired, and making use of their "*cognitive resource*" (Schoenfeld, 1985), which is the prior knowledge presented in the Development phase.
3) Review of student work (R): Teacher invites the students to present the various solutions to the original problem and leads the students to extend and generalize the problem (what is the sum of the exterior angles, what about the sum of angles of a concave polygon etc.). In teaching through problem solving, the teacher is cautious not to provide the solution of the new problem to the students, but to engage them through the problem-solving processes again. This leads to a new D-S-R cycle in the lesson. In this way, a class lesson on teaching through problem solving can be perceived of being made up of several small D-S-R cycles.

The review stage in each cycle is used by the teachers for two main purposes: (1) to consolidate student learning as each student might have approached the problem differently; (2) to facilitate the students to extend the given problem and to acquire new mathematical concepts through their solving of the generated problem. For the second purpose, differentiated instruction based on teachers' knowledge of student capability is an important factor for consideration.

## Exemplar 2: Phoney Russian Roulette problem (Toh et al., 2011)

We adapt a version of the Phoney Russian Roulette problem, that was used by Toh et al (2011) in their enactment of problem solving. *You are playing a game with your friend. Two bullets are placed in two consecutive chambers of a 6-chamber revolver. The cylinder is then spun. Your friend points the gun at their hand phone and pulls the trigger. The shot is blank. It is now your turn to point the gun at your hand phone and pull the trigger. Should you pull the trigger or spin the cylinder another time before pulling the trigger?* (p. 82)

### *Mathematical concepts and skills*

The mathematical concept involved in this problem is probability of discrete events. The problem-solving heuristics involved in solving this question involve listing and the use of diagrams and consider all the possibilities (in a reduced sample space). Although this is a question on conditional probability, the concept of conditional probability expressed explicitly could be too daunting for the general secondary school student population. Thus, the use of diagrams to consider all possible outcomes can be used as a problem-solving approach for this problem. Mathematicians have proposed the use of fundamental "*counting approach*", the more heuristic approach rather than the use of formal definition, to tackle sophisticated probability problems (Krantz, 1997).

***Instructional approach***

The mathematical concept of probability (of discrete events) forms part of the objective of the question. The teacher could either explicitly state the definition of probability and the conditions for which the definition holds or enable the students to apply their intuitive understanding of chance associated with probability. The teacher allows the students to explore the problem, raising their awareness to the use of diagrams and listing. In particular, this problem is an example of a mathematically rich problem in the stage of extending the problem. The extension of this problem lends itself into various cases which will trigger the solver to think critically in developing their probabilistic thinking.

***Required knowledge.***

The prior knowledge associated to this task is minimal for secondary school students – either the formal definition of probability or the intuitive understanding of chance as proportion of occurrence of favorable event. A more important consideration for the teacher in handling this problem is the *contextual knowledge* associated with this problem: how a revolver and its chamber functions. The contextual knowledge might form part of the scaffold for the teachers.

***Assessment***

The following are the instructional objectives which students are to achieve through attempting this problem:

1) Devise a method to systematically list the sample space of the given scenario
2) Identify elements of the sample space that belongs to the given event
3) Comparing the likelihood of two events by either using the formal definition of probability or intuitive understanding of chance by using proportion.

Note that objective (1) involves the acquisition of problem-solving heuristics of systematic listing and use of diagrams to represent all the possibilities. Objectives (2) and (3) involve engaging the students to develop understanding of the concepts related to probability.

***Complexity of problem***

In designing this problem, the teacher could anticipate to adjust the complexity of the problem by providing suitable scaffolding associated with problem and problem-specific scaffold:

1) What would you do to represent the many possible cases (i.e., draw out a diagram according to the context provided)?
2) What are the mathematical concepts you think are required to solve this problem (i.e., probability or chance, proportion)
3) Why not try to list ALL the *possible* outcomes? Which outcomes may not be possible?

***Learners' Profile***

The above problem could also be presented to students in a more lifely way, such as, acting out the problem, or using an existing or self-designed movie clip. The use of such alternative approaches of presenting a mathematics problem could address the affective need of a student and also the cognitive needs (students with reading difficulty or dyslexic). The learners' profile is a crucial determinant of the amount of cognitive load placed on students when reading the problem.

***Enactment of Lesson***

Similar to exemplar 1, the teacher could consider the D-S-R cycles in a typical lesson enactment as a guide for planning the problem-solving lesson.

1) Development (D): Teacher provides the formal definition of probability by building on their students' prior knowledge on chance and raises to the students' cognizance of mathematical problem-solving processes of systematic listing and use of diagrams. Then the teacher sets the task for the students (without providing a complete solution for the students).

2) Student work (S): Students are engaged to solve the problem using their problem-solving processes that the teachers have emphasized in the the Development phase described above.
3) Review of student work (R): Teacher invites the students to present the various solutions to the original problem, and leads the students to extend and generalize the problem (what happens if the two bullets are not in consecutive chambers? What happens if we have five bullets? instead of two bullets?). Note that not all extensions of the problem have equal level of difficulty. Here, the teacher could group the students according to their readiness.

Similar to exemplar 1, the teacher should be cautious not to provide the solution of the newly posed problem to the students, but to engage them through the problem-solving processes. This leads to a new D-S-R cycle in the lesson. In this way, a class lesson on teaching through problem solving can be perceived of being made up of several small D-S-R cycles.

## CONCLUSION

This paper aims to contribute to the ongoing effort to infuse and diffuse mathematical problem solving into the authentic mathematics classroom focusing on the secondary level. One of the efforts in enacting problem solving in the classroom is to empower the teachers to design their own task, which is one of the main concern teachers have encountered in problem solving. It is worthwhile for readers to note that the model proposed was not derived based on empirical study, but on an extensive literature review of existing mathematics education literature. Empirical study on implementing the model in the mathematics classroom may occur sometime in the near future by the researchers. Nevertheless, we hope that this study will spur further interest among mathematics educators and researchers in the area on mathematical problem solving.

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## AUTHOR'S DECLARATION

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