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An APOS analysis of preservice mathematics teachers' understanding of limits of trigonometric functions

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ABSTRACT

This paper reports on an APOS analysis of first-year undergraduate pre-service mathematics student teachers' understanding of the Sine Limit Identity (SLI) and its application in computing limits of trigonometric functions. It was a case study of sixty-eight pre-service mathematics teachers. The student teachers explored various ways of computing SLI. They also learnt how to apply the SLI in evaluating limits of other trigonometric functions. In order to determine the participants' level of understanding, the researchers analysed the participants' responses to given test items against a constructed genetic decomposition. The results of the study revealed that although more than half of the students could evaluate the sine limit, three quarters of them made some procedural, conceptual and extrapolation errors when applying the SLI in computing limits of related trigonometric functions. Based on the findings, the researchers recommended the inclusion of visual computer applications like GeoGebra as teaching tools for teaching the limits of trigonometric functions. Such applications allow students to visualise relationships among variables. The researchers also recommended further research on teaching strategies that aim at improving the teaching of the limits of trigonometric functions.

INTRODUCTION

Trigonometric functions are applicable in a myriad of situations in real life. To mention but a few, trigonometrical functions are useful in engineering, navigation, criminology and astronomy. Navigations through air and water require extensive use of trigonometry in determining directions and routes. Marine and flight engineers use trigonometry to calculate speed, direction and distance of ships and aeroplanes. Civil and mechanical engineers use trigonometric functions and ratios to calculate tilts, torques and forces on objects like bridges and buildings. In criminology, the police apply trigonometry to calculate momentum and speed of vehicles in cases of accidents. To a mathematics student, trigonometry is one of the cornerstones that link geometric, algebraic and graphical representations of mathematical statements (Nordlander, 2021).

Although in our view, deep understanding of trigonometry enhances a firm foundation for further topics in Calculus, previous research observed that concepts in trigonometry are not easy for many mathematics students (Chin, 2013; Kamber & Takaci, 2017; Kandeel, 2017; Moore & LaForest, 2014; Nurmeldina & Rafidiyah, 2019; Prabawanto & Rohimah, 2020; Orhani, 2024; Weber, 2008). According to those researchers, mathematics students often find concepts in trigonometry too abstract to understand. One of the most challenging concepts in trigonometry is 'limits of trigonometric functions' (Nordlander, 2021). However, despite the documented challenges that mathematics students face with limits of trigonometric functions, studies in this area are sparse.

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In order to enhance effective teaching of limits of trigonometric functions, there is need to examine how students understand the limit concept involving trigonometric functions. As a starting point, the current researchers thought of examining how mathematics students understand the Sine limit identity(SLI), $\lim_{\theta \to 0} \frac{\sin \theta}{\theta}$. The SLI plays a pivotal role in evaluating some more challenging limits of trigonometric functions (Nordlander, 2021; Self, 2023; Siyepu, 2015). The use of fundamental limits like the SLI helps in computing limits of trigonometric functions given in indeterminate form. The SLI is also applicable in showing mathematical proofs and in making computations involving trigonometric ratios (Farvard, 2023).

Statement of the problem

The current researchers, in their work as mathematics lecturers, witnessed poor performance by preservice mathematics teachers in determining limits of trigonometric functions. The students' poor performance triggered the current study.

Purpose of the study

The purpose of the study was to get an in-depth understanding of how preservice mathematics student teachers develop mental structures related to limits of trigonometric functions. The study focused particularly on the students' understanding of the SLI and its application in evaluating limits of other trigonometric functions. The researchers also explored the errors that the participating students made when they attempted to apply SLI in computing limits of trigonometric functions.

Research questions

The following research questions guided the study.

Q1: What level of understanding do the preservice mathematics student teachers show on the Sine limit identity and its application in solving limits of trigonometric functions?

Q2: What errors do the preservice mathematics student teachers make when computing limits of trigonometric functions involving the Sine limit identity?

Empirical studies on challenges faced by students when learning trigonometric functions

Although there are a few pedagogical researches on trigonometry, the reviewed literature revealed a number of challenges faced by students in the area. Nurmeidina and Rafidiyah (2019) studied the challenges faced by students when solving equations involving trigonometric functions. In their study, they found that students failed to solve the equations because they could not understand the mathematical statements involved. They attributed the failure by those students to lack of practice.

In a different study, Siyepu (2015) explored the errors made by mathematics students in trigonometry. Siyepu classified the errors identified into four categories, which were conceptual errors, procedural errors, extrapolation errors and interpretational errors. According to Siyepu, conceptual errors occur when a student fails to grasp a concept or fails to observe some relationship among concepts. Procedural errors occur when a student fails to apply some formulae, algorithms, rules or theories. Extrapolation errors are errors caused by over-generalization of properties, for instance in the case of limits of functions students assume that all limits of functions can be computed by direct substitution. Errors of interpretation are caused by failure by the students to get the correct meaning of mathematical statements.

Kamber and Takaci (2017) proclaim that the challenges faced by students in trigonometry at higher levels of learning start in the secondary school. According to those researchers, lack of understanding of a mathematics concept at secondary school level manifests itself at higher levels. Kandeel (2017) and Elbrink (2007) supported the notion by asserting that mathematical knowledge built atop misunderstood concepts is not likely to be successful. If the proclamation by Kamber and Takaci is anything to go by, the challenges faced by students at tertiary level in trigonometry could be a result of their failure to grasp prerequisite concepts taught in secondary school. In Zimbabwe, mathematics students learn trigonometric ratios (sine, cosine and tangent) for the first time in their third year in secondary school (MOPSE, 2015). At this level, the students learn to calculate trigonometric ratios of acute and obtuse angles. In their fifth year in secondary school, the students

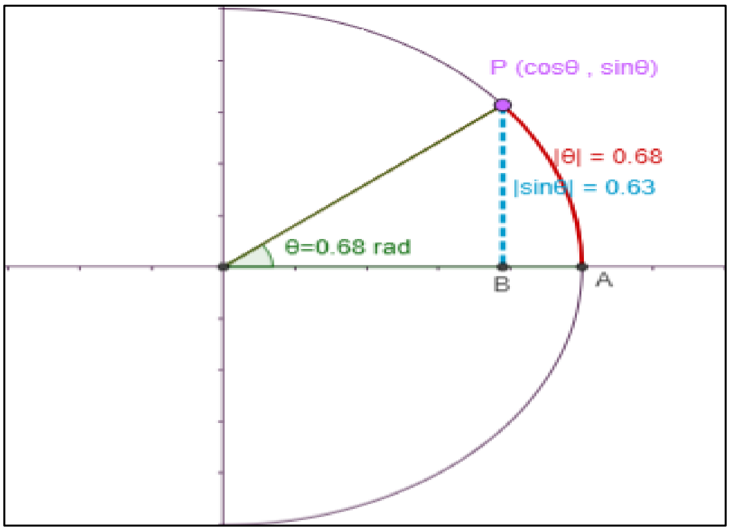

Figure 1. An extract from Geogebra application.
Source https://www.geogebra.org/m/qyzhuxdd

are introduced to trigonometric functions. The assumption is that by the time they complete secondary education, they are able to differentiate, integrate and draw graphs of simple trigonometric functions involving sine, cosine and tangent of angles measured in radians.

In an effort to mitigate the challenges faced by students in computing trigonometric limits, Sumianto (2023) suggested the use of the jigsaw-learning model. The jigsaw-learning model is a cooperative learning method where heterogeneous students are put into a group. Each member of the group is tasked to explore and understand a concept. The members then meet as a group. They share and discuss the different areas they explored. Other researchers, Nordlander (2017) as well as Baye, Ayele and Wandimuneh(2021), suggested the use of an interactive computer application called Geogebra. Nordlander's suggestion came after carrying out an experimental study with the application. In the experimental study, Nordlander analyzed how students explored $\lim_{\theta \to 0} \frac{Sin\,\theta}{\theta}$. According to the researcher, the visualizations from Geogebra assisted the students to grasp the SLI.

Figure 1 shows an illustration of how Geogebra shows the changes in the values of θ and $\sin\theta$ leading to the evaluation of $\lim_{\theta \to 0} \frac{\sin\theta}{\theta}$. When using the application, students move the point marked P along the arc. As the point P moves, the values of θ and $\sin\theta$ change accordingly. At some points, the students calculate the values of the function $\frac{\sin\theta}{\theta}$. The students observe that as the angle θ reduces in size, the function $\frac{\sin\theta}{\theta}$ approaches 1.

An analysis of the available literature shows that a number of researchers observed that concepts on trigonometry pose some challenges to many students. Different concepts on trigonometry were studied by different researchers. However, as stated earlier in this report, studies that are specifically targeted on how students learn limits of trigonometric functions are scarce. The knowledge gap found in the reviewed literature and personal experiences with mathematics students prompted the current researchers to carry out the current study.

Theoretical framework

A constructivist framework known as APOS theory guided the study. APOS theory is an improvement by Dubinsky (1984) on Piaget's genetic epistemology on how mathematics students develop mental structures as they learn mathematics concepts. APOS is an acronym whose letters stand for Action, Process, Object and Schema, which are components of mental structure development (Dubinsky, 2014; Maharaj, 2010; Salado & Trigueros, 2015).

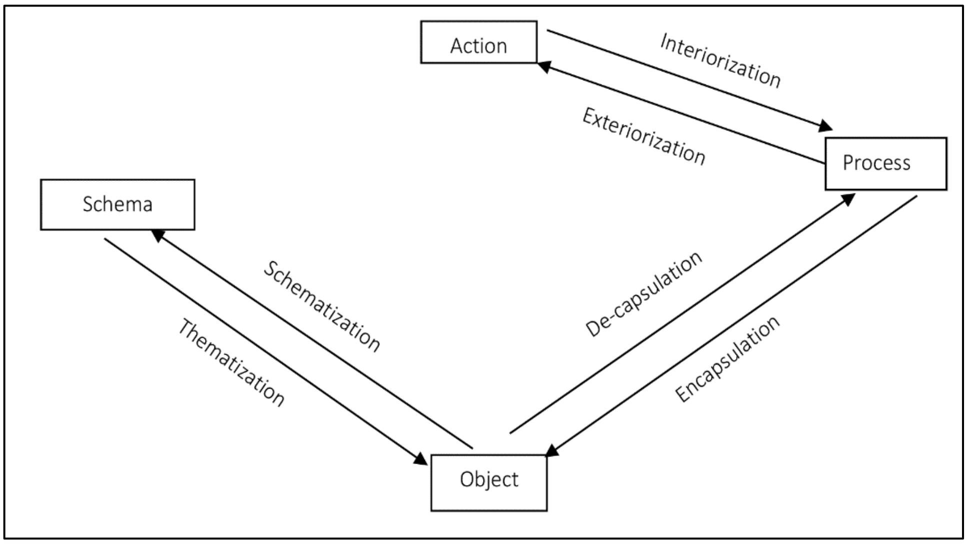

Figure 2. APOS mental structure development components (Ideas adopted from Dubinsky, 1984)

Figure 2 shows how the components of mental structure development are linked as suggested by the APOS theory. According to Weller, Arnon and Dubinsky (2011), an action is a reaction to external stimuli. It is synonymous to a situation where a person assembles a gadget by following some guidelines on a manual. Demonstrations, formula, instructions, algorithms are some of the external stimuli. The action conception forms the basis of the concept formation process. A student with an action conception can do nothing more than simply carrying out procedural computations by following a set of given instructions or imitating a demonstration. For instance, in the current study, a student with an action conception could do nothing more than computing limits of the form $\lim_{x\to 0}\frac{\sin ax}{ax}$ by simply comparing it to the SLI.

When a student repeats an action a number of times, there is a possibility that the student can interiorize the action into a mental process (Asiala et al., 2015; Dubinsky & Mcdonald, 1991). The student becomes aware of the action and can now perform it without an external stimuli or guidance(Dubinsky & Mcdonald, 2001; Soku, Okyere & Awuah, 2025). Unlike an action, a process takes place in the mind of the student and it takes place under the student's control(Dubinsky, 2014; Makonye, 2017; Van & Tong, 2022) . A student with a process conception is able to reflect, describe, reverse or combine processes. In our study, we expected a student with a process conception to have interiorised the action to evaluate limits of the form $\lim_{x\to 0}\frac{\sin ax}{ax}$ to an extent of applying the knowledge to compute limits of the form $\lim_{x\to 0}\frac{\sin ax}{bx}$.

According to Asiala et al. (1996), when a student understands the entire processes and actions involving a mathematical concept then a mental object is said to have been formed. The process of developing a mental object from a mental process is called encapsulation(Dubinsky, 1984; Mukavhi, Brijlall & Abraham, 2021). For instance, once the process and actions involved in applying SLI in computing limits of the form $\lim_{x\to 0}\frac{\sin ax}{bx}$ is encapsulated in a student's mental structure, the student is considered to have developed the $\lim_{x\to 0}\frac{\sin ax}{bx}$ mental object which helps the student to apply the SLI in solving more complex trigonometric limits. The student can carry out suitable algebraic manipulations to change the form of the given functions in order to apply the SLI. At this stage, the student perceives the SLI concept as a mental object upon which actions and processes can act. The existence of the mental object in the student's mental structure enables the student to solve problems by applying a combination of processes. The mental object can be de-capsulated back to processes and actions in the student's mental structure whenever it is necessary.

When the student interconnects objects, processes and actions to do with a particular mathematics concept, then a schema for the concept is formed (Dubinsky & Lewin, 1986; Maharaj, 2010; Tsafe, 2024). In our case, it is the schema for the SLI and its application in computing limits involving trigonometric functions. The schema comprises linked and related mental objects, processes and actions. Lower order schemata in the student's mental structure are thematized into objects for the purposes of learning higher-order concepts. In other words, lower-order schema forms a basis for higher-order schema. For example, the 'limit' schema forms the basis for the derivative schema.

In the APOS context, the students' level of conceptual understanding is analyzed by means of a tool called a genetic decomposition(GD)(Dubnisky & Lewin, 1986; Jimenez & Aguillar, 2024).The GD spells out the observable characteristics that indicate the level of mental structure development exhibited by the students. It is constructed using ideas from the researcher's own understanding of the concept, information obtained from reviewed literature and the nature of the mathematical concept (Cetin, 2019).

For the current study, the students' level of understanding was analysed using a genetic decomposition (GD) adopted from Maharaj (2010). The adopted GD was adjusted in order to suit the context of the study. According to the constructed GD, a student with an action conception was expected to at least compute limits of the form $\lim_{x\to 0}\frac{\sin ax}{ax}$ correctly.

A student with a process conception of the SLI concept and its application could show that

$$\lim_{x\to 0}\frac{\sin ax}{bx} = \lim_{x\to 0}\frac{a}{b}\frac{\sin ax}{ax}. \qquad (1)$$

A student who encapsulated the actions and mental processes involved in computing limits using SLI was expected to apply the SLI in solving more complex trigonometric limits. The student was deemed to have attained the object conception of the SLI. Such a student could change the form of the given indeterminate limits by multiplying or dividing with suitable functions and then apply the SLI to compute the given limits.

When the SLI mental object develops fully in the student's mental structure, the mental object together with the actions and the mental processes involved form the SLI schema. A student with the SLI schema could compute the limits of the given trigonometric functions by means of carrying out the necessary algebraic operations and then apply the SLI.

METHODS

Research design and the particpants

The current study followed a qualitative paradigm. It was a case study of sixty-eight preservice mathematics teachers studying for a Bachelor of Education degree at a university in Zimbabwe. This constituted the entire class of first year pre-service mathematics student teachers for the 2024 intake at the university. Calculus was a compulsory course for them. One of the concepts in their Calculus course was limits of functions, which included limits of trigonometric functions.

Ethical considerations

For purposes of anonymity, the field researcher assigned the students some numbers from one to sixty-eight. The numbers replaced the students' names. The students were informed of the study and its purpose. They were assured of the confidentiality of the information shared by them.

Data collection procedure

The students attended two-hour Calculus tutorials five times a week for four weeks. In two of the tutorials, the students learnt the SLI, $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$ and its applications in computing limits of other trigonometric functions. In one of the two tutorials, the students, with the assistance of their tutor, explored the different ways of evaluating $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}$. In the other tutorial, they learned how to apply the SLI in evaluating limits of other trigonometric functions.

The field researcher, who was also the tutor, administered a short test to the students as a way of assessing their understanding of the concept learnt. The test instructed the students to evaluate the following limits:

Table 1
Number of correct responses per test item

	Question number				
	1	2	3	4	5
Number of correct responses	51	43	65	17	23
Percentage of correct responses	75.0	63.2	95.6	25.0	33.8

1) $\lim_{x \to 0} \frac{\cos x}{x^2}$
2) $\lim_{x \to 0} \frac{\sin 3x}{4x}$
3) $\lim_{x \to 0} \frac{\sin 3x}{3x}$
4) $\lim_{x \to 0} \frac{1 - \cos x}{x \, sin \, x}$
5) $\lim_{x \to 0} \frac{\sin 3x}{\sin 4x}$

After administering the test, the field researcher interviewed the students who got the test items wrong in order to get a deep understanding of their level of understanding.

Data analysis technique

The data obtained was analysed using content analysis. In order to get the underlying themes and patterns related to the students' level of understanding, the researchers analysed the students' written responses to the given test items and the explanations they shared during interview sessions.

Validity and reliability

In order to ensure validity and reliability of the results, the researchers used instrument triangulation. Data obtained from the students' written responses to the given test items was triangulated with data obtained through interviews. Extracts from the students' written responses and verbatim statements shared by the students during interview sessions were used to support the results.

FINDINGS

Table 1 shows the number of correct responses per test item. The results show that question number 4 was the most difficult question for most of the students and question number 3 was the easiest.

Question 1

The first question required the students to compute $\lim_{x \to 0} \frac{\cos x}{x^2}$. The correct answer was $-\frac{1}{2}$. In order to compute the limit, one could apply the L' Hospital rule once and then apply the SLI. The working could proceed as shown in equation 2.

According to the GD, a student was expected to have attained at least a process conception of the SLI and its application in order to be able to identify the relationship between the given question and the SLI. Three quarters of the students got the question correct. Student 21 was one of the students who failed to provide a correct response to this question. Figure 3 shows the working provided by student 21.

Student 21 realised the need to apply the L' Hospital's rule. However, the student made a procedural error by leaving a negative sign after differentiating $\cos x$. The student went on to make another error by replacing $\sin x$ with $2 \sin 2x$. The field researcher asked the student to explain how he got $2 \sin 2x$ from $\sin x$. The student gave the following response.

"I introduced 2 to sin x so that I get 2x in the numerator," (Student 21, pers.com).

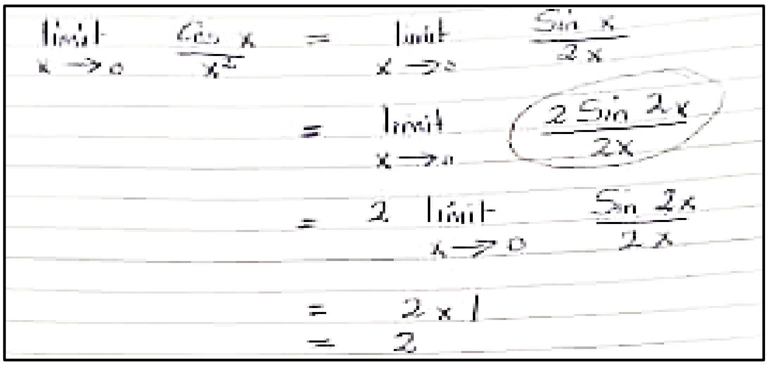

Figure 3. Response to question 1 provided by Student 21

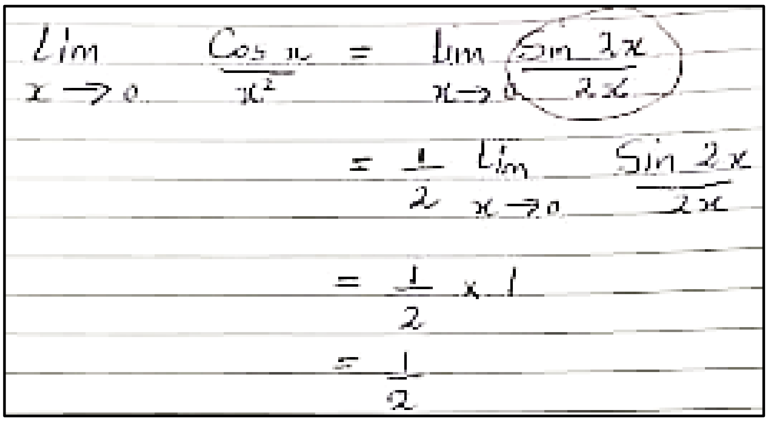

Figure 4. Response to question 1 provided by Student 22

$$\lim_{x\to 0}\frac{\cos x}{x^2}=\lim_{x\to 0}\frac{-\sin x}{2x}=\lim_{x\to 0}-\frac{1}{2}\left(\frac{\sin x}{x}\right)=-\frac{1}{2}\lim_{x\to 0}\frac{\sin x}{x}=-\frac{1}{2}.1=-\frac{1}{2}. \quad (2)$$

The last part of the working shows that the object $\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1$ had developed in the student's mental structure; however, the student had not attained the action conception of the application of the limit.

Although Student 22 got a different answer to Student 21, the two students made similar errors and the same analysis applies to both students. Student 22 failed to differentiate $Cos\ x$. During the interview session, the student failed to explain how he got $Sin\ 2x$ from $cos\ x$. Figure 4 shows the working shown by Student 22.

Question 2

The correct answer to question 2 was $\frac{3}{4}$. A possible way of computing the limit could be

$$\lim_{x\to 0}\frac{\sin\ 3x}{4x}=\lim_{x\to 0}\frac{3}{4}\left(\frac{\sin\ 3x}{3x}\right)=\frac{3}{4}\lim_{x\to 0}\frac{\sin\ 3x}{3x}=\frac{3}{4}.1=\frac{3}{4}. \quad (3)$$

About 37% of the students did not manage to provide correct responses to the question. One of those students was Student 47 who provided the working in Figure 5. Although Student 47 got the final answer correct, the student provided incorrect working. It was by coincidence that the answer was correct. The student had a misconception that $\sin(4x-x)$ was equal to $\sin 4x-\sin x$. In support of his wrong working the student gave the following explanation.

" I noticed that there was $4x$ in the denominator. To get $4x$ in the numerator, I replaced $3x$ with $4x-x$. The next step was to expand. I got $4x-\sin x$,"(Student 47,pers.com).

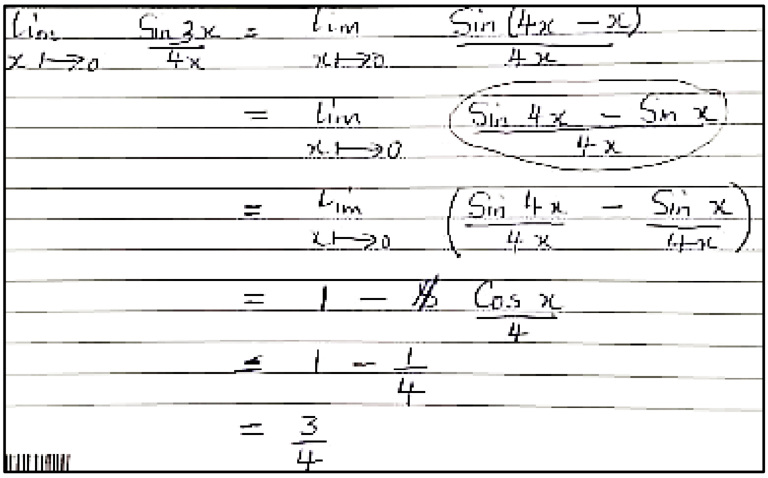

Figure 5. Response to question 2 provided by Student 47

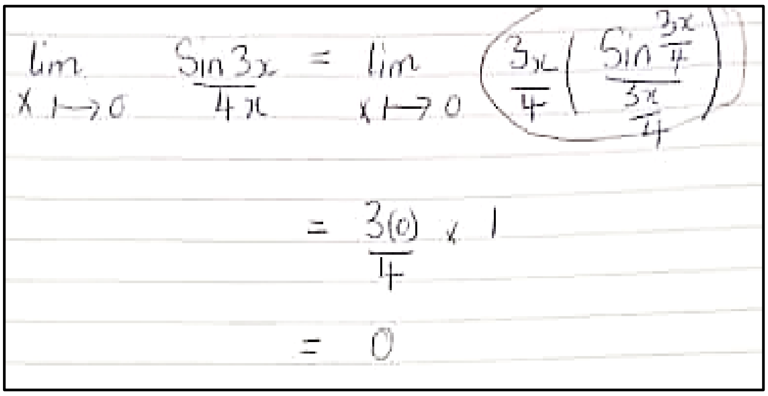

Figure 6. Response to question 2 provided by Student 17

There was evidence in Student 47's working to show that the object $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ had developed in the student's mental structure, however the student lacked the required action conception to manipulate $\frac{Sin\ 3x}{4x}$ in order to apply the SLI.

Student 17 made a serious procedural error. In an attempt to create the SLI, $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$ the student failed to factorise the given function. Like Student 47, he lacked the action and process conceptions required to manipulate the given function before applying the SLI . Figure 6 shows the working provided by Student 17.

Student 17 shared the following expalanation.

" In the first stage, I had to factor out 3x over 4 so that I have the same number inside the brackets, as you see in my working. By applying rules of limits, I got $\lim_{x \to 0} \frac{3x}{4}$ *multiplied by* $\lim_{x \to 0} \frac{sin\frac{3x}{4}}{\frac{3x}{4}}$*. The product was 0 since the other part resulted in a zero," (Student 17, pers. com).*

Question 3

About 96% of the students managed to give correct responses to question 3. The question requested the students to compute $\lim_{x \to 0} \frac{\sin 3x}{3x}$. A possible working could be

$$\lim_{x \to 0} \frac{\sin 3x}{3x} = \lim_{x \to 0} \frac{3\cos\ 3x}{3} = \cos 0 = 1. \qquad (4)$$

According to the GD, the question demanded at least an action conception. Only three students failed to provide correct responses to this question. One of those students was Students 31 who gave the response in Figure 7. Student 31 had not attained the action conception of the SLI and its

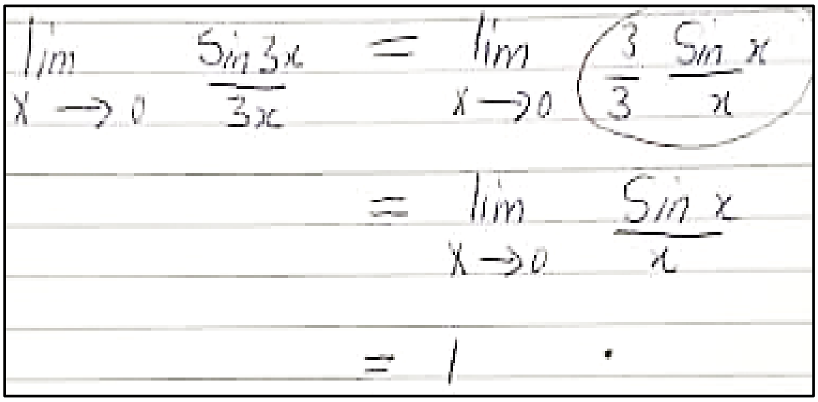

Figure 7. Response to question 3 provided by Student 31

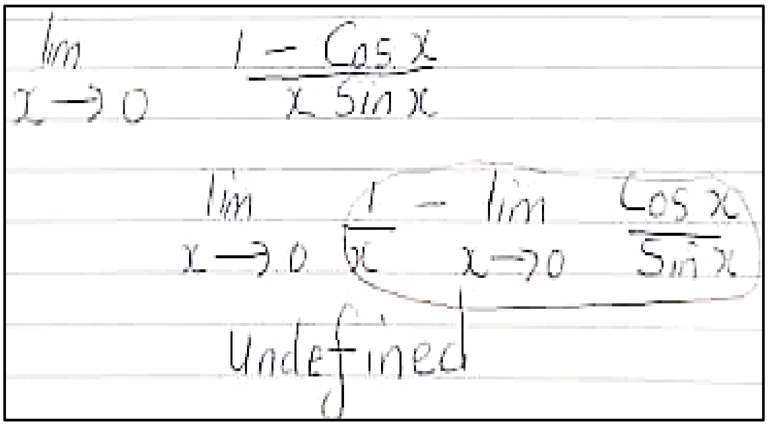

Figure 8. Response to question 4 provided by Student 54

application in solving other limits. The student failed to explain how she got $3 \sin x$ from $\sin 3x$. She had the following to say.

" Isn't it that I take out 3. Aaah I don't know,"(Student 31, pers.com).

Question 4

Question 4 appeared to be the most difficult question for more than half of the students. The question required at least an object conception of the SLI and its application. The correct answer to the question was $\frac{1}{2}$. A possible working could be:

$$\operatorname*{Lim}_{x\to 0}\frac{1-\cos x}{x\sin x}=\lim_{x\to 0}\frac{1-\cos x}{x\sin x}\left(\frac{1+\cos x}{1+\cos x}\right)=\lim_{x\to 0}\frac{1-cos^2 x}{x sin\, x}\left(\frac{1}{1+\cos x}\right)=\lim_{x\to 0}\frac{sin^2 x}{xsin\, x(1+\cos x)}=$$

$$\lim_{x\to 0}\frac{\sin x}{x}\left(\frac{1}{1+\cos x}\right)=\lim_{x\to 0}\frac{\sin x}{x}.\lim_{x\to 0}\frac{1}{1+\cos x}=1\times\frac{1}{2}=\frac{1}{2}. \quad (5)$$

Student 54 was one of the students who failed to provide a correct response to the question. The student provided the working shown in Figure 8. Student 54 attempted in vain to express $\frac{1-\cos x}{x\sin x}$ as partial fractions. The student made a serious conceptual error in the process. He did not show evidence of having attained the action conception required to solve problems of the nature given. The following excerpt was obtained from Student 54.

"………mmmmmm I am not sure but I separated the function. It then resulted in something undefined, "(Student 54, pers.com).

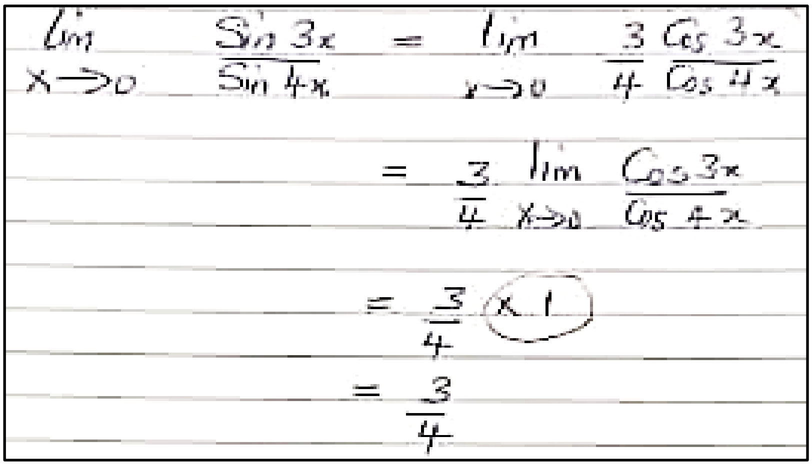

Figure 9. Response to question 5 provided by student 34

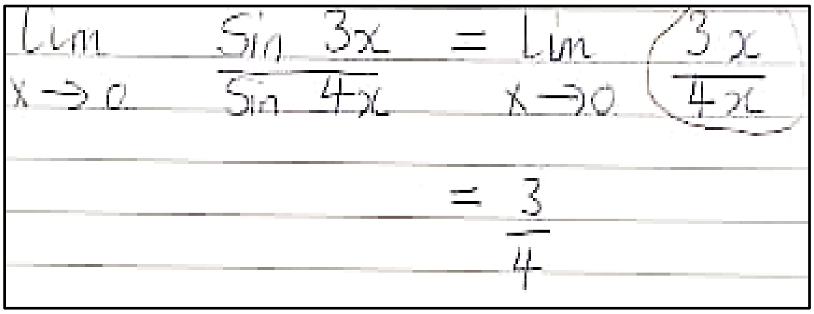

Figure 10. Response to question 5 provided by Student 19

Question 5

The fifth question of the test requested the students to compute $\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}$. The correct answer to the question was $\frac{3}{4}$. The following is a possible way of computing the limit by applying the SLI.

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 4x}=\lim_{x\to 0}\frac{3x\sin 3x}{3x}\left(\frac{4x}{4x\sin 4x}\right)=\lim_{x\to 0}\frac{3x}{4x}\left(\frac{\sin 3x}{3x}\right)\left(\frac{4x}{\sin 4x}\right)=\lim_{x\to 0}\frac{3x}{4x}.\lim_{x\to 0}\frac{\sin 3x}{3x}.\lim_{x\to 0}\left(\frac{\sin 4x}{4x}\right)^{-1}=\frac{3}{4}\times 1\times 1=\frac{3}{4} \quad (6)$$

Student 34 got the final answer to the question correct. The student applied the L' Hospital's rule correctly. However, he did not show all the parts of his working. Figure 9 shows how Student 34 proceeded with his working. When the field researcher asked Student 34 to explain the last two stages of his working, he had the following to say.

> *"cos* $3x$ *over cos* $4x$ *gives 0 over 0 when we substitute x for 0, that's where the 1 came from,"(Student 34, pers.com).*

The explanation given by Student 34 revealed that although the student got the answer correct, he had not attained the process conception of the SLI application.

Student 19 got the final answer to question 5 correct using incorrect working. She simply cancelled *sin* and x to remain with $\frac{3}{4}$. She produced the working shown in Figure 10. During the interview session, Student 19 simply said that she cancelled common factors. The student's response indicated that she had not attained the action conception of the SLI.

DISCUSSION

An analysis of the students' work revealed that some of the students had not attained the action conception of the application of the SLI. Their work had some errors. Three categories of errors stated by Siyepu (2015) were observed which were procedural, conceptual and extrapolation errors.

The procedural errors observed were mainly related to factorisation of functions and solving trigonometrical identities. For instance, a student would think that $\sin 3x$ is equal to $3 \sin x$. According to Rohimah and Prabawanto (2020) such errors were caused by students' failure to identify general comparison relationships between trigonometric functions as well as inability to carry out algebraic operations involving trigonometric functions. Obeng et al (2024) had the same view. However, in our view, failure to factorise trigonometric functions and incorrect application of trigonometric identities were the causes of the error.

The conceptual errors observed included failure to express algebraic fractions as partial fractions. There was a clear indication that some of the mathematics students had not grasped the concept of expressing algebraic fractions as partial fractions, specifically those that involve trigonometric functions. In Zimbabwe, the concept is taught in the secondary school. However, at undergraduate level some students were still unable to express given fractions as partial fractions. The failure confirms the claim by Elbrink (2007) that misconceptions developed at early stages of mathematics learning affect mathematics learning at higher levels. It is therefore important that secondary school mathematics teachers should ensure that students understand basic concepts like partial fractions and trig identities for them to be fully prepared to learn mathematics at university level.

Extrapolation errors in the form of over-generalisations were made by some of the students. Some students thought that if $\lim_{x\to 0} \frac{\sin x}{x} = 1$ then $\lim_{x\to 0} \frac{\cos x}{x}$ is also equal to 1. This misapplication of a mathematical rule would suggest that the students with such thinking had not attained the action conception of the SLI concept. In mathematics learning, however, it is common that students apply mathematical rules in wrong situations (Tatira & Mukuka, 2024).

Some of the errors made by the students were a combination of conceptual and procedural errors. For instance, one of the students equated $\frac{\sin 3x}{\sin 4x}$ to $\frac{3x}{4x}$. The student's work shows that the student had not grasped the concept of simplifying algebraic fractions involving trigonometric functions (which resulted in a conceptual error). The conceptual error resulted in the student incorrectly applying a mathematical procedure (which is a procedural error). According to Tsafe (2024), such cases where a combination of errors are made, call for mathematics teachers to be conscious of the learners' abilities so as to identify the loopholes in the learners and apply individualised learning. Sometimes a one-size-fit-all type of instruction fails to assist all students. There is need to adjust the pace, content and instruction to suit the needs of individual students, especially those who make multiple errors in their working. Various studies support the use of individualised instruction for students who lag in mathematics(Purcaru & Voinea, 2015; Smith & Johson, 2022; Wright, 2018; Zhang, Basham & Yang, 2020).

Although some of the students had problems with the application of the SLI in computing limits of other trigonometric functions, there was evidence in the working shown by the students to show that the SLI mental structure object had developed in more than half of the students. Based on the observation, the researchers made the conclusion that the teaching strategy used to evaluate the SLI was effective. The use of the visual illustrations from GeoGebra application played an important role on the effectiveness of the teaching strategy. The researchers agreed with Nordlander (2021)'s proclamation that GeoGebra illustrations reduce the abstractness of mathematical concepts by enabling students to visualise mathematical concepts.

CONCLUSION

Based on the results of the study, the researchers concluded that most of the preservice teachers had not fully developed the schema for the application of the SLI. Some of them had not fully attained the action level. The students made some procedural, conceptual and extrapolation errors that show low level of understanding of the application of the SLI concept. The researchers recommended using varying approaches when teaching the limits of trigonometric functions. Computer applications like GeoGebra are useful. They help students to visualise concepts that appear abstract to them. Limits of trigonometric functions are abstract in nature, hence there is need to find ways of reducing the abstractness. Since the results of the study revealed that some of the students who participated in the study failed to attain the action conception of the application of the SLI, the researchers recommend that studies that explore teaching strategies aimed at improving the students' understanding of trigonometric limits be carried out.

Limitation of the study

In discussing the results of the current study, the researchers took notice of the fact that the current study was a case study therefore its findings cannot be generalised. However, the findings are essential in giving an insight on how mathematics students understand the SLI and its application.

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