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Toward a Theoretical Model of Deep Mathematical Thinking: Integrating Deep Learning and Mathematical Reasoning Frameworks

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Abstract

In the context of 21st-century education, deep mathematical thinking is key to developing higher-order cognitive abilities. However, current mathematics learning predominantly focuses on procedural skills rather than conceptual understanding and reflection, often hindering the optimal development of students' deep mathematical thinking. Therefore, this study aims to develop a theoretical Deep Mathematical Thinking model through the integration of deep learning pedagogical principles and the mathematical thinking framework. Employing a systematic theoretical synthesis approach, the developed model identifies and interconnects core elements of deep learning, such as conceptual connectivity, intrinsic motivation, and metacognition with components of mathematical thinking, including reasoning, generalization, representation, and abstraction. The outcome is a three levels hierarchical Deep Mathematical Thinking model: the foundational level, the applied cognitive level, and the integrative level. This model offers theoretical and practical implications for curriculum design, assessment, and instructional practices. While still conceptual and requiring further empirical validation, the model is flexible and adaptable across diverse educational levels and contexts, positioning it as a potentially robust conceptual framework for developing reflective and meaningful mathematics learning.

Keywords: creativity, deep learning, educational model, higher-order thinking, mathematical thinking, reasoning

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1. Introduction

Mathematical thinking is an essential competency in the 21st century, encompassing complex problem-solving, logical reasoning, and knowledge transfer. This competency is crucial not only in academic settings but also in real-world contexts within the digital information era (Lehtinen et al., 2017; Schoenfeld, 1992). Without mathematical thinking skills, learners are vulnerable to difficulties in comprehending quantitative information, formulating data-based arguments, and making rational decisions (Kania et al., 2023; Yuliardi et al., 2024). Consequently, learners ideally possess proficient mathematical thinking as a core 21st-century competency (Dahlan et al., 2024). Accordingly, educators must implement mathematics pedagogy that emphasizes not only procedures but also fosters conceptual understanding, reflection, and cognitive flexibility, thereby enabling the profound and sustainable development of students' mathematical thinking (Chosya & Takiddin, 2025).

However, contemporary mathematics instructional approaches remain predominantly dominated by strategies focused on procedural memorization and algorithmic application (Liu, 2022). This tendency cultivates students as mere 'procedural followers' without grasping underlying structures or logic, resulting in superficial and non-adaptive mindsets when encountering novel or non-routine problems (Nugroho et al., 2025). Conversely, 21st-century education demands mastery of higher-order thinking skills, such as critical thinking, creativity, collaboration, and communication all grounded in mathematical reasoning (Marton & Säljö, 1997). Within this context, mathematical literacy plays a central role. Furthermore, mathematical literacy extends beyond computation to include conceptual understanding and cross-contextual application (Maryani & Widjajanti, 2020). Thus, an approach is needed that transcends proceduralism to facilitate deep conceptual understanding and encourage students to reflect on their thought processes. This aligns with the principles of pedagogical deep learning (Engel et al., 2017).

Pedagogical deep learning refers to a socio-cognitive approach where students actively construct understanding through concept integration, reflective thinking, and self-regulation characteristics congruent with deep learning principles (Entwistle & Peterson, 2004; Liu, 2022). Deep learners connect new material with prior experiences, fostering exploration, discussion, and metacognitive reflection (Maharani et al., 2024). This approach positions students as active agents who autonomously construct meaning rather than passively receive information. Thus, deep learning-based pedagogy creates opportunities for deeper cognitive engagement in mathematics education. Despite its widely acknowledged theoretical benefits, the implementation of deep learning in actual mathematics classrooms remains limited. Teaching materials and practices often provide insufficient opportunities for experimentation, discussion, or self-regulated learning (Orhani, 2024). This creates a disconnect between curriculum aspirations for meaningful understanding and the reality of instruction-centered, procedural practices.

This misalignment between theoretical ideals and practical implementation raises the question of how pedagogical deep learning principles can be effectively integrated with mathematical thinking processes. Unfortunately, literature reviews indicate that existing research is predominantly quantitative and focused on technology utilization, such as artificial intelligence. Conversely, conceptual studies explicitly linking deep learning to dimensions of mathematical thinking, such as reasoning, generalization, and abstraction remain scarce (Suglo, 2024). Yet, such research is vital for developing a unified theoretical foundation integrating both frameworks.

Additionally, extant literature reveals that pedagogical deep learning and mathematical thinking are typically investigated in isolation, with limited explicit examination of their interrelationship. Deep learning approaches are primarily developed in general contexts, such as meaningful pedagogy implementation in higher education, as explored by Biggs & Tang (2011), Entwistle & Peterson (2004), and Marton & Säljö (1976, 1997). Meanwhile, studies on mathematical thinking exemplified by Breen & O'Shea (2021) and Schoenfeld (1992, 2020) largely concentrate on reasoning and generalization within specific mathematical contexts. Consequently, no conceptual model explicitly connects core elements of deep learning (e.g., conceptual integration, reflection, intrinsic motivation) with advanced mathematical thinking components like abstraction and representation. Meta-analyses also report minimal theoretical synergy bridging these domains. The absence of an integrated theoretical framework impedes practitioners and researchers in designing pedagogical strategies or evaluative instruments that simultaneously incorporate characteristics of meaningful learning and complex mathematical thinking competencies.

Therefore, an urgent need exists to develop a conceptual model integrating deep learning perspectives, such as conceptual interconnectedness, intrinsic motivation, and metacognitive awareness into essential mathematical thinking structures. Such a model could provide a foundation for developing pedagogical strategies and assessment tools capable of evaluating deeper student understanding (Murayama et al., 2012). Accordingly, this study aims to construct a theoretical Deep Mathematical Thinking model through systematic literature synthesis. This model is expected to: (a) strengthen the conceptual basis for meaningful mathematics learning, (b) provide a theoretical framework for curriculum design and future research directions, and (c) offer evaluative indicators for assessing critical and reflective thinking in mathematics education. This research holds significant scholarly value by proposing an interdisciplinary approach that remains systematically underexplored.

Pedagogical deep learning is a holistic and reflective educational approach where learners construct understanding through integrating new knowledge with prior experiences, self-awareness, and intrinsic motivation (Grauerholz, 2001). Marton & Säljö (1976, 1997) posit that deep learners contextualize new information within broader frameworks and strive to grasp its substantive meaning rather than merely memorizing facts. Entwistle & Peterson (2004) emphasize reflection as pivotal in developing understanding, as students critically examine their thinking and learning strategies to achieve more durable conceptual comprehension. Furthermore, intrinsic motivation the inherent drive to know and understand deeply serves as the primary catalyst for learning, transcending external demands such as grades or assignments. Biggs & Tang (2011) framework links constructive elements like constructive alignment, collaboration, and reflection as foundational for fostering meaningful deep learning.

Mathematical thinking encompasses complex cognitive domains that underpin reasoning, meaning-making, and mathematical problem-solving. A core component is reasoning, which involves logical thinking, argument evaluation, and mathematical proof capabilities essential for addressing open-ended problems and ambiguous situations (Mumcu & Aktürk, 2017). Robust reasoning frequently entails discerning deeper patterns and structures, thereby catalyzing processes of generalization and abstraction. According to Breen & O'Shea (2021) and Mason et al. (2010), generalization and abstraction denote the ability to recognize patterns across individual cases and apply them to broader or theoretical contexts, hallmarks of advanced mathematical thinking.

Additionally, representation plays a critical role in facilitating mathematical thinking. The use of symbols, models, and diagrams functions not merely as visual aids but as media for constructing and reinforcing conceptual structures within mathematical cognition (Jakovác & Telcs, 2025). These dimensions are interconnected, forming an integrated cognitive framework wherein cognitive and metacognitive processes operate synergistically from problem comprehension and solution strategy design to solution verification.

2. Method

The development of the theoretical Deep Mathematical Thinking model in this study employs a theoretical synthesis approach a method designed to systematically and critically summarize, integrate, and reconceptualize existing theories (Baidoo, 2025; Salawu et al., 2023). This approach comprises three primary phases: (1) Selection and organization of core theoretical frameworks (pedagogical deep learning and mathematical thinking); (2) In-depth analysis of foundational assumptions, key concepts, and inter-construct relationships within each framework at the macro level; and (3) Reconfiguration into a cohesive, integrated conceptual model. This process demands rigorous philosophical reflection and logical justification to ensure theoretical integration is epistemologically and functionally valid, not merely rhetorical, following Jaakkola (2020) principles of theory conceptualization.

From the pedagogical deep learning perspective, key concepts include: conceptual connectivity (Marton & Säljö, 1976, 1997), intrinsic motivation (Entwistle & Peterson, 2004), and metacognition (Biggs & Tang, 2011). The mathematical thinking framework contributes elements such as reasoning, generalization, representation, and abstraction (Mason et al., 2010; Mumcu & Aktürk, 2017; Schoenfeld, 2020). Concept selection was guided by their relevance to cognitive depth (deep learning) and advanced mathematical thinking structures, with conceptual validity strengthened through meta-synthesis of meaningful mathematics learning research (Koskinen & Pitkäniemi, 2022).

The resulting theoretical model maps logical relationships between elements of both frameworks through two core mechanisms: (1) Conditional relationships denoting prerequisite-functional linkages between elements, and (2) inferential pathways explaining how one element generates another within deep mathematical thinking contexts. For instance: conceptual connectivity underpins meaningful representation formation; metacognition enables reflective abstraction processes; and intrinsic motivation drives student engagement in reasoning and generalization. This integrative logic mirrors cross-domain integration techniques in STEM frameworks (Roehrig et al., 2021), where elements interact synergistically within a unified system.

To clarify the model's structure, a three-tier hierarchical diagram visualizes components: 1) Foundational Level (conceptual connections, intrinsic motivation, metacognition), 2) Applied Cognitive Level (reasoning, generalization, representation, abstraction), and 3) Integrative Level (deep mathematical thinking processes) (see Figure 1).

This architecture resembles a pulley system in STEM 2.0 models (Roehrig et al., 2021), where components mutually support systemic equilibrium. The visualization is designed for practical application in curriculum design, assessment development, and further empirical research, aligning with blended framework approaches in instructional design (Baidoo, 2025).

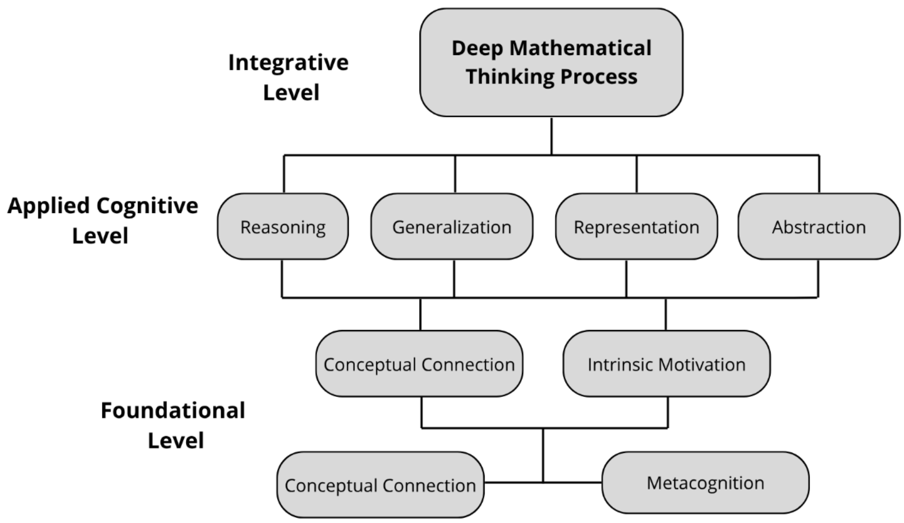

Figure 1. Hierarchical Model of Deep Mathematical Thinking

Though conceptual, the model is developed for empirical testability. Initial evaluation employs three approaches: 1) Document and content analysis (Koskinen & Pitkäniemi, 2022); 2) Expert validation by panels of mathematics education and cognition specialists; and 3) Task-based reflective mathematics interviews/written responses coded according to model components. Validation procedures adopt cognitive modeling methodologies Sun et al. (2023), emphasizing hierarchical attribute mapping with expert input.

3. Result and Discussion

a. Theoretical Implications for Instructional Design

The development of the Deep Mathematical Thinking model carries significant theoretical implications for mathematics instructional design. For decades, mathematics instruction has predominantly emphasized procedural accuracy and computational speed, often employing superficial (surface-level) and outcome-oriented approaches (Marton & Säljö, 1976). Such approaches not only constrain opportunities for developing students' critical thinking but also inhibit deep cognitive engagement with mathematical conceptual structures.

By integrating core principles from educational deep learning theory conceptual connectivity, intrinsic motivation, and metacognition mathematics pedagogy should be reoriented toward creating more reflective, meaningful, and autonomous learning experiences. Biggs & Tang (2011) emphasize that deep learning requires environments where students can connect new knowledge to prior experiences, internalize meaning through reflection, and self-regulate their learning processes. Here, the constructive alignment approach becomes essential ensuring consistent design coherence among learning objectives, activities, and assessments to optimize students' cognitive and affective engagement.

This model advocates for learning activities that transcend procedural exercises to incorporate conceptual and reflective dimensions. For instance, when teaching functions, students should not merely graph equations or compute values but also: 1) reflect on real-world meanings of representations, 2) analyze functional behavior across parameters, and 3) connect these to broader mathematical structures. This aligns with meaning-oriented instruction (Entwistle & Peterson, 2004), theoretically enhancing students' reasoning and generalization capacities.

Furthermore, intrinsic motivation a core deep learning component implies the need for personalized, context-rich, and cognitively challenging tasks. Authentic mathematical activities that foster emotional and intellectual investment can cultivate such motivation. As Trigwell & Prosser (1991) assert, personal engagement with content stimulates reflective and creative thinking patterns. Thus, teachers should design learning scenarios incorporating open-ended exploration, inquiry-based questions, and multiple solution pathways moving beyond routine exercises.

The model's metacognitive component underscores the need for explicit strategies that develop awareness of one's own thinking processes. Techniques like think-aloud protocols, self-explanation, and reflective journals during or after problem-solving activities not only help students understand their reasoning but also strengthen executive control in mathematical decision-making (Efklides, 2006).

ntegrating mathematical thinking with deep learning also necessitates assessment reform. Evaluations must extend beyond solution accuracy to examine how students construct representations, formulate generalizations, and employ reasoning. Consequently, rubric-based formative assessments accommodating mathematical thinking dimensions become vital instruments for meaning-oriented pedagogy (Schoenfeld, 2011).

Collectively, the Deep Mathematical Thinking model's theoretical implications advance a transformative framework for mathematics education prioritizing integrative, reflective, and transfer-oriented design. Instruction shifts focus from mechanistic skills or end products toward developing deep thinking processes that enable students to comprehend mathematical structures and apply them flexibly across real-world contexts. Thus, learners evolve beyond skilled problem-solvers into reflective and adaptive mathematical thinkers.

Tabel 1. Instructional Components and Strategies in the Deep Mathematical Thinking Model

Component	Theoretical Implications	Instructional Strategy Examples	Assessment Instruments
Conceptual Connectivity	Enables learners to link new mathematical ideas with prior knowledge, fostering deeper, more meaningful understanding.	– Initiate with a real-world scenario (e.g., daily temperature changes) and collaboratively build a concept map – Use group concept-mapping activities	Formative rubric evaluating depth of conceptual links
Intrinsic Motivation	Cultivates emotional and intellectual engagement by embedding math tasks in authentic, challenging contexts.	– Assign an open-ended household budget planning problem – Host small-team challenges to design innovative mathematical models	Observation checklist of participation & journal reflections
Metacognition	Develops awareness of one's own thinking processes, improving self-regulation and strategic problem solving.	– Implement think-aloud protocols during complex problems – Require reflective journals after group problem-solving sessions	Self-assessment rubric measuring quality of metacognitive reflection

Component	Theoretical Implications	Instructional Strategy Examples	Assessment Instruments
Constructive Alignment	Ensures coherence among learning objectives, activities, and assessments to optimize cognitive and affective engagement.	– Design tasks combining graphical analysis, reflective meaning-making, and conceptual generalization – Use multi-layered performance assessments	Performance rubric checking alignment of objectives, processes, and outcomes
Meaning-Oriented Instruction	Promotes deep understanding and transferability by focusing on underlying structures and real-world applicability.	– Engage students in a population-growth modeling case study – Facilitate experiments varying function parameters to observe effects	Portfolio showcasing evidence of concept transfer across contexts

Based on Table 1, Conceptual Connectivity highlights the importance of linking new mathematical ideas to students' prior knowledge. For example, before diving into functions, an instructor might begin with a discussion of daily temperature fluctuations an experience every student encounters and then collaboratively build a concept map showing how those fluctuations correspond to the shape of a function's graph. Such activities strengthen the relational network among ideas, and a formative rubric can gauge how deeply learners integrate new concepts with their existing mental frameworks.

The Intrinsic Motivation component seeks to foster students' emotional and intellectual engagement by presenting context-rich, challenging tasks. As an illustration, learners might tackle an open-ended budgeting problem for a household requiring them to apply arithmetic and algebra in a real-world scenario or participate in small-team competitions to design innovative mathematical models. Throughout these tasks, the teacher observes levels of participation and reviews students' reflective journal entries to assess the emergence of genuine, self-driven motivation.

Metacognition centers on developing students' awareness of their own thinking processes. Techniques such as think-aloud protocols during complex problem solving allow both teacher and peers to hear the student's reasoning steps, while reflective journals written after group discussions encourage learners to document and evaluate the strategies they employed. A self-assessment rubric then provides a systematic way to measure the quality of each student's metacognitive reflections, such as their ability to spot logical errors or plan subsequent steps.

With Constructive Alignment, every element of the instructional design objectives, activities, and assessments is deliberately aligned to optimize both cognitive and affective engagement. In a lesson on functions, this means crafting tasks that require graphical analysis, meaning-making reflections, and conceptual generalizations, all paired with a multi-layered performance assessment. Clear rubrics guide the teacher in verifying that the learning objectives, the student processes, and the outcomes are all in sync.

Finally, Meaning-Oriented Instruction emphasizes deep understanding and the capacity to transfer knowledge to new contexts. For instance, students might work on an integrated case study modeling population growth, then experiment with different function parameters to see firsthand how each change affects the outcome. A portfolio containing evidence of their work across varied situations serves as an effective assessment tool, revealing the extent to which learners can apply mathematical concepts flexibly in real-world problems.

b. Implications for Future Research

The Deep Mathematical Thinking model developed in this study offers not only a conceptual contribution but also opens avenues for multidimensional future research agendas. Implications for subsequent research include the need for empirical validation, cross-context exploration, evaluation instrument development, and studies on the roles of teachers and technology in model implementation. Given the theoretical and integrative nature of this model, the primary challenge for future research lies in operationalizing, measuring, and testing its constituent elements within diverse authentic mathematics learning contexts.

Without a strong empirical foundation, this model risks becoming merely a normative theoretical framework that is difficult to operationalize in real practice. Therefore, testing in real classroom situations becomes an important step to ensure that every element in the model truly reflects the cognitive dynamics of students during the mathematics learning process.

Future research should prioritize empirical testing of inter-element relationships within the model. Although constructed from a conceptual synthesis of deep learning theory (Entwistle & Peterson, 2004; Marton & Säljö, 1976) and mathematical thinking frameworks (Lithner, 2008; Schoenfeld, 2011), the interconnections among these elements lack empirical verification. Subsequent studies could examine relationships between conceptual connectivity, intrinsic motivation, and metacognition relative to students' reasoning, generalization, and abstraction abilities. This could be achieved through developing valid and reliable measurement instruments, such as scales, observational protocols, or mathematical thinking process analyses.

Such research may employ design-based research (DBR) or mixed-methods approaches. DBR provides space to develop learning interventions based on this model and test them through iterative classroom cycles (Wang & Hannafin, 2011), enabling contextual and dynamic refinement rather than static validation. Meanwhile, mixed methods allow triangulation between quantitative data (e.g., student motivation or metacognition) and qualitative data (e.g., visual representations or reasoning strategies employed in problem-solving).

Further research should also investigate contextual and cultural dimensions of model implementation. Cross-cultural or cross-educational system studies would reveal how this model functions within distinct learning frameworks. In Indonesia's transitioning curriculum shifting toward differentiated, competency-based learning opportunities exist to test deep learning elements amid structural educational challenges. Kember (2000) research demonstrates that cultural learning orientations significantly influence students' tendencies toward deep or surface learning approaches.

Another critical implication involves developing novel measurement instruments and indicators to capture latent dimensions of the model, such as mental representation construction or metacognitive strategies. Measuring these dimensions requires methods that are not merely descriptive but also exploratory and interpretative. Relevant approaches include stimulated recall, verbal protocol analysis, and cognitive task analysis (Ericsson & Simon, 1993), which capture thought processes not always directly observable. Developing rubrics, interview protocols, observation sheets, and learning analytics-based digital applications constitutes vital future methodological directions.

As educational technology evolves, the integration of learning analytics and adaptive learning platforms can be an effective means of tracking the development of mathematical thinking in real-time, while also providing automated feedback tailored to each student's cognitive profile.

Moreover, teacher professional development emerges as an indispensable factor. While teachers are key actors in implementing meaningful learning, few studies explicitly connect teacher understanding of deep learning theory and mathematical thinking. Qualitative phenomenological or case study research could explore how teachers interpret and design mathematics instruction based on this model, aligning with Mellati et al. (2015) findings that teachers' pedagogical beliefs and knowledge substantially impact student learning experiences. Teacher competency strengthening should not only encompass technical training for model implementation but also involve the reconstruction of perspectives toward mathematics learning as a dialogical, reflective, and meaningful process, rather than merely the transmission of procedures and rules.

Additionally, longitudinal studies tracking long-term development of deep mathematical thinking are essential. Unlike procedural skills, reasoning and abstraction abilities develop gradually through complex processes. Long-term research would deepen understanding of how deep thinking processes form, evolve, or regress across students' academic trajectories. Such findings are crucial for informing sustainable curriculum policies and pedagogical interventions. Long-term research can also provide insights into how deep mathematical thinking skills correlate with academic success across subjects, as well as life skills such as data-driven decision-making, complex problem-solving, and quantitative literacy.

Finally, this model enables development of more contextualized and specialized variants. Researchers could create sub-models for specific educational levels (e.g., elementary, middle, high school), mathematical domains (e.g., algebra, geometry, statistics), or student groups (e.g., gifted learners or those with learning difficulties). This flexibility positions the Deep Mathematical Thinking model as an initial conceptual framework adaptable to diverse mathematics education research initiatives. Thus, the model serves not merely as an endpoint of conceptual synthesis but as a promising springboard for further academic exploration in developing meaningful mathematics learning theory, methodology, and practice.

Tabel 2. Proposed Future Research Agenda for the Deep Mathematical Thinking Model

Research Focus	Key Questions	Methods / Instruments	Expected Outcomes
Empirical Validation	– To what extent do the model's elements reflect students' actual cognitive processes? – How are the elements interrelated?	– Field testing in real classrooms – Measurement scales & observation protocols – Statistical analysis of variable relationships	– Empirical evidence for model validity – Mapping of each element's strengths and weaknesses
Inter-Element Relationships	– Do conceptual connectivity, intrinsic motivation, and metacognition influence reasoning abilities?	– Reliable measures for each variable (surveys, verbal protocols) – Path analysis	– Quantitative understanding of correlations and causal links among model elements

Research Focus	Key Questions	Methods / Instruments	Expected Outcomes
DBR & Mixed-Methods	– How can the model be refined through iterative classroom cycles? – How do quantitative and qualitative data complement each other?	– Design-Based Research (Wang & Hannafin, 2011) – Surveys, interviews, classroom observations	– Prototype learning interventions based on the model – Continuous improvement recommendations
Contextual & Cultural Dimensions	– How does the model function across different educational systems and cultures? – What are the implications within Indonesia's competency-based curriculum?	– Cross-cultural / cross-school comparative studies – Analysis of cultural learning orientations (Kember, 2000)	– Adaptation mapping by cultural context – Implementation guidelines for varied settings
Novel Measurement Instruments	– How can we measure mental representations and metacognitive strategies? – Which tools are truly exploratory and interpretative?	– Stimulated recall, verbal protocol, cognitive task analysis (Ericsson & Simon, 1993) – Rubrics, observation sheets, LA apps	– Valid and reliable instrument set for in-depth research – Sample rubrics and interview/observation protocols
Technology & Learning Analytics	– How can learning analytics and adaptive platforms track and provide real-time feedback? – What impact do they have on development of mathematical thinking?	– Development of an LA dashboard – Experimentation with adaptive learning platforms	– Prototype cognitive-progress monitoring system – Data on feedback efficiency and effectiveness for students
Teacher Professional Development	– How do teachers' understandings of deep learning theory translate into practice? – What role do pedagogical beliefs play?	– Phenomenological / case studies (Mellati et al., 2015) – In-depth interviews and focus groups	– CPD training model incorporating perspective reconstruction – Reflective practice training modules
Longitudinal Studies	– How do students' reasoning and abstraction skills evolve over time? – How are these skills linked to academic success and life skills?	– Multi-year longitudinal research – Periodic assessments & student portfolio tracking	– Trajectory charts of deep thinking development – Evidence to inform sustainable curriculum and policy
Contextualized Model Variants	– How can sub-models be adapted for specific grades or domains (algebra, geometry, statistics)? – How to tailor for special student groups?	– Development and validation of context-specific sub-models – Pilot tests with targeted samples	– Detailed framework variants for each level/domain – Adaptation guidelines for gifted learners and those with learning difficulties

Based on Table 1, several key directions emerge for scholars aiming to deepen both theoretical and practical understanding of the model. First, empirical validation and investigation of inter-element relationships are foundational. Field testing in real classrooms, combined with robust measurement scales, observation protocols, and statistical techniques such as path analysis, will establish whether the model's components conceptual connectivity, intrinsic motivation, and metacognition truly reflect students' cognitive dynamics and how they causally interact to support reasoning, generalization, and abstraction.

Next, methodological innovation is paramount. Design-Based Research (DBR) offers an iterative, context-sensitive approach to refine learning interventions, while mixed-methods designs will triangulate quantitative indicators (e.g., motivation scores, metacognitive self-reports) with qualitative data (e.g., verbal protocol analyses, classroom discourse) to yield a richer, more nuanced picture of how deep mathematical thinking unfolds. Equally important is the development of novel instruments stimulated recall, cognitive task analyses, rubrics, and learning-analytics dashboards that go beyond descriptive measures to capture latent dimensions like mental representation construction and real-time feedback loops.

Finally, contextual and longitudinal perspectives will ensure the model's adaptability and sustainability. Comparative studies across cultures and educational systems including Indonesia's evolving competency-based curriculum will map how local learning orientations shape model implementation. Teacher professional development, explored through phenomenological or case-study approaches, will reveal how educators' beliefs and pedagogical practices mediate student engagement with deep thinking processes. Longitudinal research, spanning multiple academic years, will chart trajectories of reasoning and abstraction skills, informing curriculum policies and revealing connections to broader life-long competencies. Moreover, creating specialized sub-models for different grade levels, mathematical domains, and learner populations will position the Deep Mathematical Thinking model as a flexible framework ready for diverse educational research and practice.

c. Strengths and Limitations of the Model

1) Strengths of the Model

The Deep Mathematical Thinking model offers significant theoretical and practical advantages, positioning it as an innovative framework for designing meaningful and profound mathematics learning. Its strengths lie in its multidisciplinary synthesis, educational applicability, and flexibility across diverse learning levels and contexts.

a) Systematic Integration of Two Theoretical Disciplines

The model's primary strength is its successful integration of two major theoretical approaches that have historically developed in parallel: deep learning theory in education (Biggs & Tang, 2011; Entwistle & Peterson, 2004) and mathematical thinking theory (Lithner, 2008; Schoenfeld, 2011). This integration is systematically achieved through a three-tiered hierarchical structure for deep mathematical thinking processes: (1) the foundational level encompassing intrinsic motivation, metacognition, and conceptual connectivity; (2) the applied cognitive level involving reasoning, generalization, representation, and abstraction; and (3) the integrative level as the pinnacle of deep thinking. Such an integrative approach is crucial in modern education as it encourages synergy between general learning theories and domain-specific needs, resulting in pedagogical models that are more contextual and functional in real classroom practice.

While deep learning theory has primarily been developed in general higher education contexts without domain-specific focus, mathematical thinking research has largely remained within mathematics education, seldom explicitly linked to reflective learning processes. By unifying these approaches, the Deep Mathematical Thinking model provides a comprehensive cross-disciplinary synthesis that bridges general cognitive theory and mathematics-specific pedagogy.

b) Practicality: Translates to Curriculum Design and Observational Instruments

A key advantage is the model's high applicability. Its hierarchical structure enables its use as a framework for designing learning activities, formative assessments, and classroom observation tools. For instance, at the foundational level, teachers can design activities fostering motivation and self-reflection, such as learning journals, reflective discussions, or concept mapping. At the applied cognitive level, educators can develop exploratory tasks promoting diverse representations and generalization abilities.

In classroom research contexts, the model provides a basis for developing rubrics or coding systems to identify students' thinking levels during mathematical problem-solving. As Panizzon (2003) emphasizes, explicit conceptual frameworks are essential for detecting variations in students' cognitive levels a need directly addressed by this model's structure.

1. Flexibility Across Educational Levels and Classroom Contexts

The model demonstrates notable flexibility, being adaptable from primary to higher education. While thinking complexity varies across levels, core elements, such as reasoning, representation, and metacognition can be identified and developed at all cognitive stages (Vosniadou, 2001). For example, conceptual connections and reasoning in primary education can be cultivated through concrete visual activities, while abstract generalization and proof in higher education can be facilitated via symbolic discourse and theoretical exploration.

Furthermore, the model is pedagogy-agnostic. It can be implemented across approaches like Problem-Based Learning (PBL), Realistic Mathematics Education (RME), and other constructivist methods. This allows teachers to integrate the framework into contextually responsive instructional designs aligned with student needs. The model's flexibility also allows for cross-curricular and cross-cultural adaptation, making it highly relevant for implementation in diverse educational systems, such as in the context of Indonesia's Merdeka Curriculum, which emphasizes differentiated and contextual learning.

2) Limitations of the Model

Despite its conceptual and pedagogical contributions, the Deep Mathematical Thinking model has several limitations regarding practical implementation and theoretical scope. Acknowledging these constraints is essential for future refinement.

a) Conceptual Nature Lacking Empirical Validation

The primary limitation is the model's conceptual status. Although grounded in a rigorous synthesis of theories and prior research, it lacks empirical validation in authentic learning contexts. Thus, its effectiveness in enhancing students' mathematical thinking and its implementability by teachers remain unproven through direct evidence.

Empirical validation is critical, particularly given the model's complexity, which demands nuanced measurement tools and layered research designs. As Akker et al. (2007) assert, conceptual models only yield significant contributions when systematically operationalized and tested within actual instructional settings.

b) Underemphasized Affective and Socio-Cultural Dimensions

The model insufficiently addresses affective and socio-cultural factors crucial to mathematics learning. Notably absent is explicit consideration of mathematics anxiety, proven to inhibit reflective thinking and effective metacognitive strategy use (Marsh & Dolan, 2007). Such affective states influence students' representation construction, reasoning approaches, and interpretation of errors as learning opportunities.

Additionally, the model does not explicitly integrate sociocultural perspectives. Yet social interactions, learner identity, and mathematical language are vital for constructing deep mathematical meaning. As Boaler (2002) emphasizes, mathematics learning practices are shaped by classroom norms, teacher perceptions, and cultural values. Thus, expanding the framework to incorporate sociocultural perspectives would enhance its relevance and sensitivity to complex social contexts.

c) Implementation Complexity in Time-Constrained Classrooms

The hierarchical structure spanning cognitive and affective dimensions requires deep educator understanding and meticulous lesson planning. Implementation may prove challenging in time-constrained classrooms facing curriculum pressures and high administrative burdens. Teachers unfamiliar with reflective approaches or untrained in meaningful mathematics pedagogy may struggle to design activities holistically addressing all model elements.

Furthermore, pressures to meet national assessment targets often prioritize procedural mastery and immediate outcomes over deep thinking processes. As Murray (2013) explains, successful pedagogical reform hinges on teachers' professional capacity and systemic support enabling innovation within rigid educational structures.

Tabel 3. Summary of Strengths and Limitations of the Deep Mathematical Thinking Model

Category	Aspect	Description
Strength	– Systematic Integration of Two Theoretical Disciplines	– Combines deep learning theory in education (Biggs & Tang, 2011; Entwistle & Peterson, 2004) with mathematical thinking theory (Lithner, 2008; Schoenfeld, 2011) into a three-tiered hierarchy (foundational, applied cognitive, integrative), bridging general cognitive theory and math-specific pedagogy.
	– Practicality: Curriculum & Observational Instruments	– Provides a clear framework for designing learning activities (e.g., journals, concept maps), formative assessments, and classroom observation tools or rubrics, enabling teachers and researchers to identify and foster different levels of students' mathematical thinking in real classrooms.
	– Flexibility Across Levels & Contexts	– Adaptable from primary through higher education and pedagogy-agnostic (e.g., PBL, RME), supporting concrete visual tasks for young learners up to abstract proof work in university, and capable of cross-cultural and cross-curricular use (e.g., Indonesia's Merdeka Curriculum).
Limitation	– Conceptual Nature Without Empirical Validation	– Remains a theoretically grounded but untested framework; its real-world effectiveness, measurement tools, and teacher implementability have not yet been demonstrated through systematic empirical studies in authentic classroom settings.

Category	Aspect	Description
	– Underemphasized Affective & Socio-Cultural Dimensions	– Does not explicitly address mathematics anxiety or other affective factors, nor sociocultural influences such as learner identity, social interaction, and cultural norms, all of which critically shape how students engage in representation, reasoning, and reflection.
	– Implementation Complexity in Time-Constrained Classrooms	– Its multi-layered structure demands deep teacher understanding and detailed lesson planning, posing challenges under heavy curriculum pressures, limited class time, and high-stakes assessments especially where professional support for reflective, meaningful mathematics pedagogy is lacking.

Based on Table 1, the Deep Mathematical Thinking model demonstrates three core strengths and three principal limitations. First, its systematic integration of deep learning theory in education with mathematical thinking theory creates a cohesive, three-tiered hierarchy that bridges general cognitive frameworks and domain-specific pedagogy, enabling more contextualized and functional classroom practice. Second, the model's practicality lies in its clear guidance for curriculum design and observational instruments teachers can draw on its levels to create motivational activities, reflective journals, concept maps, and rubrics that pinpoint students' thinking during problem solving. Third, the model offers flexibility: it applies from primary through higher education, across pedagogical approaches such as Problem-Based Learning or Realistic Mathematics Education, and is adaptable to various cultural or curricular contexts like Indonesia's Merdeka Curriculum.

However, Table 1 also highlights key limitations. Its conceptual nature means it remains untested in authentic classrooms; without empirical validation, its impact on student learning and the feasibility of teacher implementation are unknown. The model also underemphasizes affective and socio-cultural dimensions, omitting factors such as mathematics anxiety and the role of social interaction, identity, and cultural norms that critically influence how students reason and reflect. Finally, its implementation complexity can overwhelm teachers working under tight time constraints, heavy assessment pressures, and limited professional support, making comprehensive adoption challenging without systemic training and support.

4. Conclusion

This study proposes the conceptual Deep Mathematical Thinking model as an integrative framework bridging deep learning theory in education with mathematical thinking theory in mathematics. The model represents a synthesis of three hierarchical levels: (1) the foundational level (intrinsic motivation, metacognition, and conceptual connectivity); (2) the applied cognitive level (reasoning, generalization, representation, and abstraction); and (3) the integrative level as the culmination of profound and reflective mathematical thinking. Through this structure, the model offers a cross-disciplinary contribution that expands perspectives on designing meaningful learning to cultivate advanced mathematical understanding.

The model's implications encompass applications in curriculum development, assessment design, instructional planning, and future research. Educators and curriculum developers may utilize this framework to design activities and evaluations that stimulate students' conceptual understanding and reflective thinking. Furthermore, the model creates avenues for empirical research through structural validity testing and classroom implementation studies. Given its high flexibility, the Deep Mathematical Thinking model holds potential for adaptation not only within mathematics education but also across disciplinary contexts emphasizing the integration of cognitive reasoning and deep reflection.

This model not only presents a comprehensive theoretical construct but also provides a systematic foundation for transforming mathematics learning from a procedural nature into a meaningful, reflective, and contextual learning experience. In the era of 21st-century education, which demands mathematical literacy, critical thinking, and knowledge transfer abilities, this model emerges as a strategic alternative capable of addressing the challenges of modern curricula. Thus, Deep Mathematical Thinking not only serves as a result of theoretical synthesis but also as a potential instrument for creating transformative pedagogical change, oriented towards developing students' holistic, adaptive, and visionary competencies.

The Deep Mathematical Thinking model developed by Santosa et al. (2021) is an innovative framework that unites a profound understanding of learning and mathematical thinking. Although still theoretical, this model holds great potential for reforming mathematics learning to be more reflective, contextual, and meaningful. Further research, especially classroom-based and with valid instruments, is highly needed to test and develop this model.

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